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On the Two-Dimensional Stochastic Ising Model in the Phase Coexistence Region Near the Critical Point

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We consider the two-dimensional stochastic Ising model in finite square Λ with free boundary conditions, at inverse temperature $\beta > \beta_c$ and zero external field. Using duality and recent results of Ioffe on the Wulff construction close to the critical temperature, we extend some of the results obtained by Martinelli in the low-temperature regime to any temperature below the critical one. In particular we show that the gap in the spectrum of the generator of the dynamics goes to zero in the thermodynamic limit as an exponential of the side length of Λ, with a rate constant determined by the surface tension along one of the coordinate axes. We also extend to the same range of temperatures the result due to Shlosman on the equilibrium large deviations of the magnetization with free boundary conditions.

KEY WORDS: Stochastic Ising model; phase coexistence; relaxation time; spectral gap; surface tension; large deviations.

INTRODUCTION

In the last two years there has been considerable progress in the detailed description of the two-dimensional kinetic Ising model inside and outside the phase coexistence region.

When the inverse temperature β and the external field h are such that there is no phase transition, it has been finally established that the relaxation

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time (which in our discussion we take as the inverse of the gap in the spectrum of the generator of the dynamics) of any Glauber-type dynamics in a finite rectangle is bounded from above uniformly in the size of the rectangle and in the boundary conditions. Such a bound was first proved for $\beta < \beta_c$ and arbitrary h or $h \neq 0$ and $\beta \gg \beta_c$ (see refs. 15, 13, and 14) and only recently it has been extended to any $\beta > \beta_c$ and arbitrary $h \neq 0$.[19] Here β_c denotes the critical point.

On the other hand, when $h = 0$ and $\beta > \beta_c$, the infinite-volume Glauber dynamics is no longer ergodic due to the existence of two extremal Gibbs states and one easily realizes that the relaxation time in an $L \times L$ square with free boundary conditions diverges when $L \to \infty$. In ref. 11 (see Theorem 4.2) the precise logarithmic asymptotics of the gap was computed for $\beta \gg \beta_c$ and found to be

$$\text{gap}(L, \text{free boundary conditions }) \sim \exp[-\beta\tau_b(0)\, L]$$

where $\tau_\beta(0)$ stands for the surface tension in the direction of one of the coordinate axes. A similar behavior has recently been proved in ref. 6 for other boundary conditions that, roughly speaking, do not favor either one of the two phases, without, however, a precise estimate of the rate constant.

The picture of the relaxation to the Gibbs equilibrium measure that comes out of the results of ref. 11 and subsequently substantially improved in ref. 12, is the following: the system first relaxes rather rapidly to one of the two phases and then it creates, via a large fluctuation, a thin layer of the opposite phase along one of the sides of Λ. Such a process requires already a time of the order of $\exp^{[\beta\tau_\beta(0)\, L]}$. After that, the opposite phase invades the whole system by moving, in a much shorter time scale, the interface to the side opposite to the initial one and equilibrium is finally reached. Once this picture is established it is not difficult to show that, under a suitable stretching of the time by a factor $a(L) \approx \exp^{[\beta\tau_\beta(0)\, L]}$, the magnetization in the square Λ behaves in time as a continuous Markov chain with state space $\{-m^*, +m^*\}$ and unitary jump rates, where m^* is the spontaneous magnetization (see Theorem 6.1 of ref. 11).

The key ingredients of the results in ref. 11 were as follows:

(i) A geometric bound on the gap in the spectrum of the generator of the dynamics in a rectangle.

(ii) Some detailed equilibrium estimate on the fluctuations of a horizontal interface of length L.

(iii) A precise estimate of the equilibrium probability of having anomalous magnetization $m_\Lambda(\sigma) \in (-m^*, m^*)$.

The first was borrowed from a clever technique to bound from below the spectral gap of symmetric Markov chains on complicated graphs introduced in ref. 9 in the framework of hard computational problems. It is important for us that its validity is independent of β. The second and third were available at low enough temperature thanks to the results of refs. 2 and 17 on the rigorous Wulff construction. Here the requirement of large β was quite essential since most of the results in ref. 2 were obtained through cluster expansions methods.

Recently, however, there has been remarkable progress in the rigorous theory of the Wulff construction for *any* $\beta > \beta_c$, using the powerful methods of duality.[7,8] Duality had already played a relevant role in the approach to the Wulff construction and phase separation developed in ref. 16 for the 2D Ising model with + boundary conditions at low temperature and the main contribution of the above two papers was to combine it with the Fortuin–Kasteleyn representation of the Ising model to push some of the most relevant results of ref. 16 up to the critical point excluded. The ideas and techniques developed in these two papers were also fundamental to the analysis of metastability for the 2D Ising model very close to the line of first-order phase transition.[21] Relevant simplifications of the techniques of refs. 7 and 8 can be found in refs. 19 and 20.

In this paper we use duality techniques to extend up to β_c the basic estimates (ii) and (iii) above. As a consequence we compute for any $\beta > \beta_c$ the logarithmic asymptotics of the relaxation time in the thermodynamic limit and show that it has the same form as the one computed in ref. 11. Our result, however, is stated in a slightly weaker form than the corresponding one in ref. 11 since we do not give any explicit bounds on the finite-volume correction. This is due to the fact that, contrary to what happens in the low-temperature regime, where the powerful methods of cluster expansion are available, duality methods so far seem able to provide good estimates on the fluctuations of an interface only when these are of the same order of magnitude as the length of the interface itself.

Once we have (ii) for any $\beta > \beta_c$, it follows from the same technique described in Section 3 of ref. 11 that if we replace on one side of the square Λ the free boundary conditions with *plus* b.c., then the relaxation time of the new system becomes much smaller than it was before and in particular it can be bounded from above by an exponential in L with a rate that vanishes in the thermodynamic limit. Using this result, one could extend up to β_c the results of Section 6 of ref. 11 on the Markov chain description of the magnetization mentioned above. We decided, however, to skip this part in order to avoid too much repetition.

We conclude this introduction with a brief summary of the various sections. In Section 1 we fix the notation and state the major result. In

Section 2 we recall the main steps of the argument given in ref. 11 and prove the main theorem assuming three key propositions. In Section 3, the core of the paper, we prove the main technical estimates. In Section 4 we use the results of Section 3 to give the proof of two of the three propositions used in Section 2. In Section 5 we prove the last proposition of Section 2 by extending to any $\beta > \beta_c$ the result of ref. 17 on the large deviations of the magnetization with free boundary conditions.

1. NOTATION AND RESULTS

1.1. General Definitions

We consider the two-dimensional lattice $\mathbb{Z}^2$, whose elements are called *sites*, and its dual $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$. For $x, y \in \mathbb{R}^2$ we define the distances

$$d(x, y) = |x - y| = \sum_{i=1}^{2} |x_i - y_i|$$

$$d_2(x, y) = |x - y|_2 = \left(\sum_{i=1}^{2} |x_i - y_i|^2 \right)^{1/2}$$

$[x, y]$ is the *closed segment* with x, y as its endpoints. The *edges* of $\mathbb{Z}^2(\mathbb{Z}^2_*)$ are those $e = [x, y]$ with x, y nearest neighbors in $\mathbb{Z}^2(\mathbb{Z}^2_*)$. Given an edge e of $\mathbb{Z}^2$, e^* is the unique edge in $\mathbb{Z}^2_*$ that intersects e. The *boundary of an edge* $e = [x, y]$ is $\delta e = \{x, y\}$. The *boundary of a subset of edges* α is the set of sites $\delta\alpha$ that belong to an odd number of edges of α. A set of edges is called *closed* if its boundary is empty. We denote by $\mathscr{E}_\Lambda$ the set of all edges such that both endpoints are in Λ and by $\bar{\mathscr{E}}_\Lambda$ the set of all edges with at least one endpoint in Λ. Conversely, for a set of edges X, $\mathscr{V}(X)$ stands for the set of all sites which are endpoints of at least one edge in X.

Given $\Lambda \subset \mathbb{Z}^2$, we let $\Lambda^c = \mathbb{Z}^2 \backslash \Lambda$ and define Λ^* as the set of all $x \in \mathbb{Z}^2_*$ such that $d(x, \Lambda) = 1$. The set of the dual edges is defined as

$$\mathscr{E}^*_\Lambda = \{e^* : e \in \bar{\mathscr{E}}_\Lambda\}$$

Notice that, in general, $\mathscr{E}^*_\Lambda \subset \mathscr{E}_{\Lambda^*}$ (the equality holds, for instance, in the case of rectangles). We will often consider our model on a $(2L+1) \times (2M+1)$ rectangle

$$Q_{L,M} = \{(x_1, x_2) \in \mathbb{Z}^2 : -L \leqslant x_1 \leqslant L,\ -M \leqslant x_2 \leqslant M\}$$

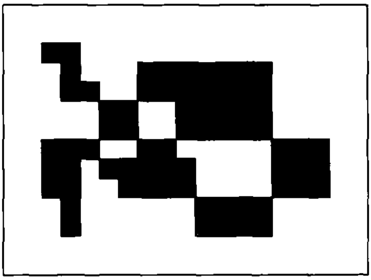

Fig. 1. The interior of a closed set of dual edges.

Q_L stands for $Q_{L,L}$. If Λ is finite, we write $\Lambda \subset\subset \mathbb{Z}^2$. The cardinality of Λ is denoted by $|\Lambda|$. We define the *boundaries*

$$\partial\Lambda = \{x \in \Lambda : d(x, \Lambda^c) = 1\}$$

$$\partial^+\Lambda = \{x \in \Lambda^c : d(x, \Lambda) = 1\}$$

$$\delta\Lambda = \{e^* : e \in \bar{\mathscr{E}}_\Lambda \backslash \mathscr{E}_\Lambda\}$$

$(x_1,..., x_n)$ is called a *path* from x_1 to x_n if $|x_{i+1} - x_i| = 1$ for $i = 1,..., n-1$. A $*$-path is the same as a path with $|x_{i+1} - x_i| = 1$ replaced by $|x_i - x_{i+1}|_2 \in \{1, \sqrt{2}\}$. A ($*$-) path is called *self-avoiding* if $x_i \neq x_j$ for all $\{i, j\}$ such that $i \neq j$ and $\{i, j\} \neq \{1, n\}$. If $x_1 = x_n$ the ($*$-) path is called *closed*.

We say that $\Lambda \subset \mathbb{Z}^2$ is *connected* ($*$-connected) if for all x, y in Λ there exists a path ($*$-path) from x to y which is entirely contained in Λ. We call $\Lambda \subset\subset \mathbb{Z}^2$ *simply connected* if Λ^c is connected. A set of edges α is *connected* if the union of all its edges is connected in $\mathbb{R}^2$.

If $\alpha \subset \mathscr{E}^*_{\mathbb{Z}^2}$ is a finite, closed set of dual edges, then we define the *interior* of α as the set of all sites $x = (x_1, x_2) \in \mathbb{Z}^2$ such that the half-line

$$\{x_1\} \times [x_2, +\infty)$$

intersects α in an odd number of points (see Fig. 1).

The interior of α is denoted by int α.

1.2. The Ising Model

We consider the standard 2D Ising model with *configuration space* $\Omega = \{-1, +1\}^{\mathbb{Z}^2}$, or $\Omega_V = \{-1, +1\}^V$ for some $V \subset \mathbb{Z}^2$. An element of Ω_V will usually be denoted by $\sigma = \{\sigma(x), x \in V\}$. If $U \subset V \subset \mathbb{Z}^2$ and $\sigma \in \Omega_V$, we denote by σ_U the restriction of σ to the set U. If U, V are disjoint, $\sigma_U \eta_V$

is the configuration on $U \cup V$ which is equal to (...guess what) σ on U and η on V.

Given $V \subset\subset \mathbb{Z}^2$ and some *boundary condition* (b.c.) $\eta \in \Omega$, we consider the Hamiltonian

$$-H_V^{J,\eta}(\sigma) = \frac{1}{2} \sum_{\substack{x,\, y \in V \\ |x-y|=1}} J(x, y)(\sigma(x)\,\sigma(y) - 1)$$

$$+ \sum_{\substack{x \in V,\, y \in V^c \\ |x-y|=1}} J(x, y)(\sigma(x)\,\eta(y) - 1) \tag{1.1}$$

The coupling J has been introduced for technical reasons, but our main result is for $J = 1$. We always assume $0 \leqslant J(x, y)$ for all x, y.

For further convenience we define the Hamiltonian with *free boundary conditions* using a different normalization

$$-H_V^{J,\varnothing}(\sigma) = \frac{1}{2} \sum_{\substack{x,\, y \in V \\ |x-y|=1}} J(x, y)\,\sigma(x)\,\sigma(y) \tag{1.2}$$

The partition function is given by

$$Z^{\beta, J, \eta}(V) = \sum_{\sigma \in \Omega_V} \exp[-\beta H_V^{J,\eta}(\sigma)] \tag{1.3}$$

When $J(x, y) = 1$ for all x, y, we drop the superscript J. We sometimes consider also the Hamiltonian with free b.c. on a restricted set of bonds: If $X \subset \mathscr{E}_V$, we write

$$-H_{V,X}^{J,\varnothing}(\sigma) = \sum_{[x,\, y] \in X} J(x, y)\,\sigma(x)\,\sigma(y) \tag{1.4}$$

Of course this is equivalent to having $J(x, y) = 0$ unless $[x, y] \in X$. The partition function corresponding to (1.4) is denoted by $Z^{\beta, J, \varnothing}(V, X)$.

When V is a rectangle $V = Q_{L,M}$, we define the $[k]$ boundary condition by [let $x = (x_1, x_2)$]

$$[k](x) = \begin{cases} -1 & \text{if } \; x_2 \geqslant M - k + 1 \\ +1 & \text{if } \; x_2 \leqslant M - k \end{cases} \tag{1.5}$$

so, in particular, $[0]$ b.c. means -1 on the top side of the rectangle and $+1$ on the remaining three sides. A rectangle V has a δ-boundary δV

consisting of a top, bottom, left, and right sides, which we denote, respectively, with $\delta_t V$, $\delta_b V$, $\delta_l V$, and $\delta_r V$. So, for instance,

$$\delta_t Q_{L,M} = \{e = [x, y]^* : [x, y] = [(j, M), (j, M+1)] \; j = -L,..., L\}$$

In the following we will often choose $J = 1$ everywhere with the exception of one or more sides of a certain rectangle, where we take $J = \varepsilon$. So we introduce the notation

$$\begin{aligned} J_\varepsilon^{\square}(V; x, y) &= \begin{cases} \varepsilon & \text{if } [x, y]^* \in \delta V \\ 1 & \text{otherwise} \end{cases} \\ J_\varepsilon^{||}(V; x, y) &= \begin{cases} \varepsilon & \text{if } [x, y]^* \in \delta_l V \cup \delta_r V \\ 1 & \text{otherwise} \end{cases} \\ J_\varepsilon^{\sqcup}(V; x, y) &= \begin{cases} \varepsilon & \text{if } [x, y]^* \in \delta V \backslash \delta_t V \\ 1 & \text{otherwise} \end{cases} \\ J_\varepsilon^{\sqcap}(V; x, y) &= \begin{cases} \varepsilon & \text{if } [x, y]^* \in \delta V \backslash \delta_b V \\ 1 & \text{otherwise} \end{cases} \end{aligned} \tag{1.6}$$

The (finite-volume) conditional Gibbs measure associated with the Hamiltonian (1.1) is defined as

$$\mu_V^{\beta, J, \tau}(\sigma) = \begin{cases} (Z^{\beta, J, \tau}(V))^{-1} \exp[-\beta H_V^{J, \tau}(\sigma)] \\ \quad \text{if } \sigma(x) = \tau(x) \text{ for all } x \in V^c \\ 0 \quad \text{otherwise} \end{cases} \tag{1.7}$$

The expectation with respect to the measure (1.7) is denoted by $\mathbf{E}_V^{\beta, J, \tau}(\cdot)$. The set of measures (1.7) satisfies the DLR compatibility conditions

$$\mu_\Lambda^{\beta, J, \tau}(\sigma) = \sum_{\sigma' \in \Omega} \mu_\Lambda^{\beta, J, \tau}(\sigma') \, \mu_V^{\beta, J, \sigma'}(\sigma) \qquad \forall V \subset \Lambda \subset\subset \mathbb{Z}^2 \tag{1.8}$$

One introduces a partial order on Ω_V by saying that $\sigma \leqslant \sigma'$ if $\sigma(x) \leqslant \sigma'(x)$ for all $x \in V$. A function $f: \Omega_V \mapsto \mathbb{R}$ is called *monotone increasing* (*decreasing*) if $\sigma \leqslant \sigma'$ implies $f(\sigma) \leqslant f(\sigma')(f(\sigma) \geqslant f(\sigma'))$. An event is called *positive* (*negative*) if its characteristic function is increasing (decreasing). Given two probability measures μ, μ' on Ω_V, we write $\mu \leqslant \mu'$ if $\mu(f) \leqslant \mu'(f)$ for all increasing functions f [by $\mu(f)$ we denote the expectation with respect to μ].

In the following we will take advantage of the FKG inequalities,[4] which state that:

(1) If $\eta \leqslant \eta'$, then $\mu_V^{\beta, J, \eta} \leqslant \mu_V^{\beta, J, \eta'}$.

(2) If f and g are increasing, then $\mathbf{E}_V^{\beta, J, \tau}(fg) \geqslant \mathbf{E}_V^{\beta, J, \tau}(f)\, \mathbf{E}_V^{\beta, J, \tau}(g)$.

The Surface Tension. We denote by $\tau_\beta(\vartheta)$ the surface tension at angle ϑ $(0 \leqslant \vartheta \leqslant \pi/2)$, which measures the free energy of an interface in the direction orthogonal to the vector $n_\vartheta = (\cos \vartheta, \sin \vartheta)$. We refer the reader to ref. 16 for a precise definition. In the standard Ising model we have, by symmetry,

$$\tau_\beta(\vartheta) = \tau_\beta(\vartheta + \pi/2) = \tau_\beta(\pi/2 - \vartheta) \tag{1.9}$$

We extend τ_β to a function on $\mathbb{R}^2$ by setting

$$\tau_\beta(x) = |x|_2\, \tau_\beta\left(\frac{x}{|x|_2}\right)$$

We also let for simplicity

$$\tau_\beta = \tau_\beta(\vartheta = 0) = \tau_\beta(\vartheta = \pi/2)$$

For the reader's convenience we recall (see Lemma 6.3 in ref. 16) that

$$\mathbf{E}_\Lambda^{\beta, \varnothing}\sigma(x)\, \sigma(y) \leqslant \exp[-\beta^* \tau_{\beta^*}(x-y)] \tag{1.10}$$

where β^* is the dual value for β [see (1.19)]. We are also going to make use of the so-called sharp triangle inequality (Lemma 2.1 of ref. 7), which in particular implies that if $\beta > \beta_c$, there exists $D(\beta) > 0$ such that

$$\tau_\beta(\vartheta) \geqslant \tau_\beta \cos \vartheta + D(\beta) \sin^2(\vartheta/2) \tag{1.11}$$

As a consequence of (1.11) and (1.9) we have

$$\tau_\beta(\vartheta) \geqslant \frac{\tau_\beta}{\sqrt{2}} \tag{1.12}$$

For further relevant properties of the surface tension the reader is again referred to ref. 16.

1.3. Contours

We use the contour representation with the so-called *splitting rules* (see, for instance, ref. 2). Given (σ, η, Λ), we define $\mathscr{B}_\Lambda^\eta(\sigma)$ as the set of all unsatisfied edges in $\mathscr{E}_\Lambda^*$ of the configuration σ subject to b.c. η, i.e., if $\rho = \sigma_\Lambda \eta_{\Lambda^c}$, then

$$\mathscr{B}_\Lambda^\eta(\sigma) = \{e^* = [x, y]^* \in \mathscr{E}_\Lambda^* : \rho(x) \neq \rho(y)\}$$

If the b.c. η is $+$ (or $-$), then $\mathscr{B}_\Lambda^\eta(\sigma)$ is closed, while in general it has a nonempty boundary. We have the following result.

Proposition 1.1. If $\Lambda \subset\subset \mathbb{Z}^2$, $\eta \in \Omega$, then:

(i) The boundary of $\mathscr{B}_\Lambda^\eta(\sigma)$ does not depend on σ, so in particular $\delta\mathscr{B}_\Lambda^\eta(\sigma) = \delta\mathscr{B}_\Lambda^\eta(+1)$.

(ii) $\delta\mathscr{B}_\Lambda^\eta(\sigma) \subset \Lambda^*$.

(iii) If Λ is simply connected, $\mathscr{B}_\Lambda^\eta$ is a one-to-one mapping from Ω_Λ onto the set

$$\{X \subset \mathscr{E}_\Lambda^* : \delta X = \delta\mathscr{B}_\Lambda^\eta(+1)\}$$

Parts (i) and (ii) are straightforward. For (iii) see, for instance, Lemma 6.1 in ref. 12, which deals with the case $\eta = +1$ (the generalization is easy).

It is useful to decompose $\mathscr{B}_\Lambda^\eta(\sigma)$ and in general an arbitrary set X of dual edges as a collection of *contours* γ_i

$$X = \gamma_1 \cup \cdots \cup \gamma_n \tag{1.13}$$

which have the advantage that they can be associated with simple self-avoiding (open or closed) curves in $\mathbb{R}^2$. Decomposition (1.13) is intuitively obtained by cutting all three- and four-edge meetings along the southwest to northeast direction. More precisely, we first consider the subset of $\mathbb{R}^2$,

$$\hat{X} = \bigcup_{e \in X} e$$

Then, at each dual site where three or four edges of X meet, we operate the *rounding the corner* operation shown in Fig. 2.

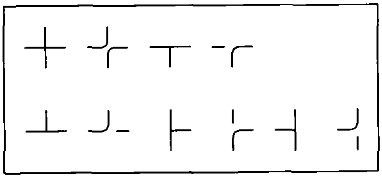

Fig. 2. The splitting rules.

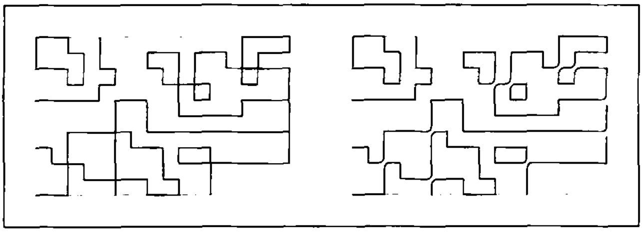

Fig. 3. An example of operating the splitting rules.

In this way we obtain another subset $\tilde{X}$ of $\mathbb{R}^2$ whose Hausdorff distance from $\hat{X}$ is less than, say, 1/3. Now $\tilde{X}$ can be decomposed as a union of its connected components $\tilde{X} = Y_1 \cup \cdots \cup Y_n$. Finally, we define γ_i as the set of all edges which "belong" to Y_i, i.e.,

$$\gamma_i = \{e \in X\text{: the Hausdorff distance of } e \text{ from } Y_i \text{ is less than } 1/3\}$$

Figure 3 shows a set of edges on the left and the corresponding collection of contours on the right.

We now give an independent definition of contours and we introduce the notion of *compatibility* between contours in such a way as to have a one-to-one correspondence between sets of edges and compatible collections of contours.

Let us denote by $\hat{e}_n$, $\hat{e}_s$, $\hat{e}_e$, and $\hat{e}_w$ the unit vectors in $\mathbb{R}^2$ pointing in directions respectively north, south, east, and west. Let also $\hat{e}_n(x)$ be the edge whose endpoints are x and $x + \hat{e}_n$ (and similarly for the other directions). To each edge e we associate a set of three edges $\Delta(e)$ given by

$$\Delta(\hat{e}_e(x)) = \hat{e}_e(x) \cup \hat{e}_s(x) \cup \hat{e}_n(x + \hat{e}_e)$$

$$\Delta(\hat{e}_n(x)) = \hat{e}_n(x) \cup \hat{e}_w(x) \cup \hat{e}_e(x + \hat{e}_n)$$

The elements in $\Delta(e)$ are said to be *forbidden* by e (for reasons which will become clear when the "compatibility" is introduced). We extend the notion of forbidden edges to contours by setting

$$\Delta(\gamma) = \bigcup_{e \in \gamma} \Delta(e)$$

The definition of $\Delta(e)$ is such that, for any two edges e, e' we have

$$\Delta(e) \ni e' \qquad \text{if and only if} \quad e \in \Delta(e')$$

Given a set E of edges (or dual edges), a *contour* in E is a finite sequence of elements of E, $\gamma = (e_1,..., e_n)$, such that it is possible to write $e_i = [x_i, y_i]$ and (1) all edges e_i are distinct, (2) $y_i = x_{i+1}$, and (3) for any two nonconsecutive edges e, e' we have $e \notin \Delta(e')$.

If $x_1 = y_n$, the contour is called *closed*, otherwise it is *open*. We do not distinguish between two contours with the same set of edges. The representation $(e_1,..., e_n)$ of γ is thus always to be interpreted modulo a reversal of the ordering of the edges and, for closed contours, modulo a cyclic permutation. Two edges e and e' are said to be *consecutive* in a contour γ if they are consecutive in at least one representation.

We will sometimes use the same symbol γ to denote the set $\{e_1,..., e_n\}$ and we will improperly write things like $e \in \gamma$ and $\gamma \subset X$.

For further convenience we introduce a modified version of $\Delta(e)$. If $x \in \delta e$, we let

$$\Delta(e, x) = \{e' \in \Delta(e): x \notin \delta e'\} \cup e$$

and for all open contours $\gamma = (e_1,..., e_n)$ with $e_i = [x_{i-1}, x_i]$,

$$\begin{aligned} \Delta'(\gamma, x_0) &= \bigcup_{i=2}^{n} \Delta(e_i) \cup \Delta(e_1, x_0) \\ \Delta'(\gamma, x_n) &= \bigcup_{i=1}^{n-1} \Delta(e_i) \cup \Delta(e_n, x_n) \\ \Delta''(\gamma) &= \bigcup_{i=2}^{n-1} \Delta(e_i) \cup \Delta(e_1, x_0) \cup \Delta(e_n, x_n) \end{aligned} \tag{1.14}$$

Sometimes it is more convenient to consider the forbidden sites, so we let

$$\Delta^s(\gamma) = \{x \in \mathbb{Z}^2: d_2(x, \Delta(\gamma)) = 1/2\} \tag{1.15}$$

The *boundary* $\delta\gamma$ of a contour is given by the usual boundary of γ when γ is thought of as a set of edges. Thus $\delta\gamma$ can either be empty or consist of a pair of sites. Given a collection of contours $\underline{\gamma} = \{\gamma\}$, its *boundary* $\delta\underline{\gamma}$ is defined as

$$\delta\underline{\gamma} = \bigcup_{\gamma \in \underline{\gamma}} \delta\gamma$$

(notice that the boundary of a collection of contours is not equal to the boundary of the set of all edges in some $\gamma \in \underline{\gamma}$). Given a set of edges X, we let

$$C(X) = \{\gamma: \gamma \text{ is a closed contour in } X\}$$

$$C(X, \{x, y\}) = \{\gamma: \gamma \text{ is a contour in } X \text{ with boundary } \delta\gamma = \{x, y\}\}$$

Two contours γ_1 and γ_2 are said to be *compatible* if their boundaries are disjoint and

$$\Delta(\gamma_1) \cap \gamma_2 = \varnothing \qquad \text{or equivalently} \qquad \gamma_1 \cap \Delta(\gamma_2) = \varnothing$$

A collection $\underline{\gamma}$ of contours in X is called *compatible* if each pair in $\underline{\gamma}$ is compatible. We define

$$C^*(X, U) = \text{the set of all compatible collections } \underline{\gamma} \text{ of contours in } X \text{ such that } \delta\underline{\gamma} = U$$

For simplicity $C^*(X) = C^*(X, \varnothing)$ denotes compatible collections of closed contours.

The *interior* of a compatible collection of closed contours $\underline{\gamma}$ is defined as

$$\operatorname{int} \underline{\gamma} = \operatorname{int} \left[\bigcup_{\gamma \in \underline{\gamma}} \gamma \right] \tag{1.16}$$

(the interior of a closed set of edges was defined in Section 1.1).

Is is straightforward to check that there is a one-to-one correspondence between $C^*(X, U)$ and the set of all subsets $Y \subset X$ such that $\delta Y = U$. The collection of contours corresponding to the set of edges $\mathscr{B}''_\Lambda(\sigma)$ will be denoted by $\mathscr{G}''_\Lambda(\sigma)$.

We introduce now the *contour partition function*, which is a basic object in what follows. Let then $X \subset\subset \mathscr{E}^*_{\mathbb{Z}^2}$, $U \subset\subset \mathbb{Z}^2_*$ and let $\underline{\lambda}$ be a compatible collection of contours in X whose boundary is contained in U, i.e., $\underline{\lambda} \in C^*(X, V)$ for some $V \subset U$. We define

$$\tilde{Z}^{\beta, J}(X, U; \underline{\lambda}) = \sum_{\underline{\gamma}:\, \underline{\gamma} \cup \underline{\lambda} \in C^*(X, U)} w_{\beta, J}(\underline{\gamma})$$

where the *weight* of a collection of contours is given by

$$w_{\beta, J}(\underline{\gamma}) = \prod_{\gamma \in \underline{\gamma}} w_{\beta, J}(\gamma) = \prod_{\gamma \in \underline{\gamma}} \prod_{e \in \gamma} \exp[-2\beta J(e)]$$

If the arguments U or $\underline{\lambda}$ are missing in $\tilde{Z}(X, U; \underline{\lambda})$, they must be interpreted as the empty set, so in particular $\tilde{Z}(X)$ is the sum of the weights of all compatible collections of closed contours in X. The notion of compatibility between contours is such that we have

$$\tilde{Z}^{\beta, J}(X, U; \underline{\lambda}) = \tilde{Z}^{\beta, J}(X \backslash \Delta(\underline{\lambda}), U \backslash \delta\underline{\lambda})$$

If $\Lambda \subset\subset \mathbb{Z}^2$ is simply connected, we can express the usual partition function as a contour partition function by

$$Z^{\beta, J, \eta}(\Lambda) = \tilde{Z}^{\beta, J}(\mathscr{E}_\Lambda^*, \delta\mathscr{B}_\Lambda^\eta(+1))$$

In the special case that Λ is a rectangle, $\Lambda = Q_{L,M}$, with $[k]$ b.c. [see (1.5)], then an open contour will appear on going from a_k to b_k, where a_k and b_k are the two extreme sites in the $(k+1)$th row (from the top) of Λ^*, i.e.,

$$a_k = (-L-\tfrac{1}{2}, M+\tfrac{1}{2}-k), \qquad b_k = (+L+\tfrac{1}{2}, M+\tfrac{1}{2}-k) \tag{1.17}$$

and the partition function is given by

$$Z^{\beta, J, [k]}(\Lambda) = \tilde{Z}^{\beta, J}(\mathscr{E}_\Lambda^*, \{a_k, b_k\}) = \sum_{\gamma \in C(\mathscr{E}_\Lambda^*, \{a_k, b_k\})} w_{\beta, J}(\gamma)\, \tilde{Z}^{\beta, J}(\mathscr{E}_\Lambda^*; \gamma)$$

Thanks to the duality properties of the 2D Ising model,[16] one can also write the partition function with free b.c. as

$$Z^{\beta, J, \varnothing}(\Lambda, X) = 2^{|\Lambda|} \prod_{e \in X} \cosh[\beta J(e)]\, \tilde{Z}^{\beta^*, J^*}(X) \qquad \text{for any} \quad X \subset \mathscr{E}_\Lambda \tag{1.18}$$

where β^* and J^* are determined by the duality relationships

$$e^{-2\beta} = \tanh \beta^*, \qquad e^{-2\beta J(e)} = \tanh[\beta^* J(e)^*] \tag{1.19}$$

Equation (1.18) implies

$$Z^{\beta, J, +}(\Lambda) = 2^{-|\Lambda^*|} \prod_{e \in \mathscr{E}_\Lambda^*} \{\cosh[\beta^* J(e)^*]\}^{-1} Z^{\beta^*, J^*, \varnothing}(\Lambda^*, \mathscr{E}_\Lambda^*) \tag{1.20}$$

The critical value β_c is determined by the equality $\beta_c = \beta_c^*$. Expectations of products of spin variables can be expressed as quotients of contour partition functions[16]

$$\mathbf{E}_{\Lambda, X}^{\beta, J, \varnothing}\left(\prod_{x \in U} \sigma(x)\right) = \frac{\tilde{Z}^{\beta^*, J^*}(X, U)}{\tilde{Z}^{\beta^*, J^*}(X)} \tag{1.21}$$

where $\mathbf{E}_{\Lambda, X}^{\beta, J, \varnothing}$ is the expectation with respect to the Gibbs measure associated to the Hamiltonian (1.4).

We anticipate a result which will be useful in the following.

Proposition 1.2. Let X, Y be two finite sets of dual bonds such that $X \subset Y$ and let

$$F(J) = \frac{\tilde{Z}^{\beta, J}(X)}{\tilde{Z}^{\beta, J}(Y)}$$

Then $F(J)$ is nondecreasing in each variable $J(x, y)$ such that $[x, y]^* \notin Y \backslash X$.

Proof. Let Λ be any finite set of sites such that $Y \subset \mathscr{E}^*_\Lambda$. Then, by (1.18)

$$F(J) = \frac{Z^{\beta^*, J^*, \varnothing}(\Lambda, X)}{Z^{\beta^*, J^*, \varnothing}(\Lambda, Y)} \prod_{e = [x, y]^* \in Y \backslash X} \cosh[\beta^* J^*(x, y)]$$

Thus, if $e \notin Y \backslash X$, we have

$$\frac{1}{\beta^*} \frac{d}{dJ^*(x, y)} \log F(J) = \mathbf{E}^{\beta^*, J^*, \varnothing}_{\Lambda, X} \sigma(x)\, \sigma(y) - \mathbf{E}^{\beta^*, J^*, \varnothing}_{\Lambda, Y} \sigma(x)\, \sigma(y)$$

which is nonpositive by the second Griffiths inequality. ∎

Contours with Free Boundary Conditions. A configuration σ in Λ with free boundary conditions produces a collection of contours $\underline{\gamma} = \mathscr{G}^{\varnothing}_\Lambda(\sigma)$ such that:

(i) $\underline{\gamma} \subset \mathscr{E}^*_\Lambda \backslash \delta\Lambda$.

(ii) Each contour $\gamma \in \underline{\gamma}$ is either closed or open with its boundary $\delta\gamma \subset \mathscr{V}^{\cdot}(\delta\Lambda)$.

So we let $\hat{\mathscr{E}}_\Lambda = \mathscr{E}^*_\Lambda \backslash \delta\Lambda$ and for an arbitrary set of dual edges X and any $U \subset\subset \mathbb{Z}^2_*$ we define

$$C(X, \subset U) = \bigcup_{V \subset U} C(X, V), \qquad C^*(X, \subset U) = \bigcup_{V \subset U} C^*(X, V)$$

There is a two-to-one mapping $\underline{\gamma} \mapsto \underline{\gamma}'$ from $C^*(\mathscr{E}^*_\Lambda)$ to $C^*(\hat{\mathscr{E}}_\Lambda, \subset \mathscr{V}^{\cdot}(\delta\Lambda))$ given by

$$\underline{\gamma}' = \left(\bigcup_{\gamma \in \underline{\gamma}} \gamma \right) \Big\backslash \delta\Lambda \tag{1.22}$$

where it is understood that the LHS of (1.22) must be partitioned in the proper way according to the splitting rules. In this way we can write, for each simply connected Λ,

$$Z^{\beta, \varnothing}(\Lambda) = 2 e^{\beta |cE_\Lambda|} \sum_{\underline{\gamma} \in C^*(\hat{\mathscr{E}}_\Lambda, \subset \mathscr{V}^{\cdot}(\delta\Lambda))} w(\underline{\gamma})$$

The following proposition will be used later in the paper.

Proposition 1.3. Let Λ be a rectangle, $\sigma \in \Omega_\Lambda$, and let γ be a contour in $\mathscr{G}_\Lambda^{\varnothing}(\sigma)$. Then:

(i) If γ is open, there exist $\lambda' \in C(\mathscr{E}_\Lambda^* \backslash \Delta''(\gamma), \delta\gamma)$ and $\lambda \in C(\mathscr{E}_\Lambda^*)$ such that $\lambda = \gamma \cup \lambda'$ and $\lambda \in \mathscr{G}_\Lambda^+(\sigma)$.

(ii) If γ is closed, then $\gamma \in \mathscr{G}_\Lambda^+(\sigma)$.

The proof is more or less straightforward and we omit it. We just observe that the result is false for arbitrary shapes of Λ.

1.4. The Dynamics and Our Result

The stochastic dynamics we want to study is defined by the Markov generator

$$(L_V^\eta f)(\sigma) = \sum_{x \in V} c(x, \sigma)[f(\sigma^x) - f(\sigma)] \tag{1.23}$$

(in this subsection we consider β and J fixed and we mostly omit them) acting on $L^2(\Omega, d\mu_V^\eta)$, where

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ -\sigma(y) & \text{if } y = x \end{cases}$$

In (1.23) σ denotes a configuration on the whole lattice $\mathbb{Z}^2$ which, in view of (1.7), agrees with the b.c. η on V^c. In general we identify $L^2(\Omega, d\mu_V^\eta)$ with $L^2(\Omega_V, d\mu_V^\eta)$. The nonnegative real quantities

$$c(x, \sigma), \quad x \in \mathbb{Z}^2, \quad \sigma \in \Omega$$

are the *transition rates* for the process.

The assumptions on the transition rates are:

(H_1) *Nearest neighbor interactions.* If $\sigma(y) = \sigma'(y)$ for all y such that $d(x, y) \leqslant 1$, then $c(x, \sigma) = c(x, \sigma')$.

(H_2) *Attractivity.* If $\sigma \leqslant \sigma'$ and $\sigma(x) = \sigma'(x)$, then

$$\sigma(x)\, c(x, \sigma) \geqslant \sigma'(x)\, c(x, \sigma') \tag{1.24}$$

(H_3) *Detailed balance*:

$$\exp[-\beta H_{\{x\}}^\sigma(\sigma(x))]\, c(x, \sigma) = \exp[-\beta H_{\{x\}}^\sigma(-\sigma(x))]\, c(x, \sigma^x) \tag{1.25}$$

(H$_4$) *Positivity and boundedness.* There exist $c_m(\beta)$ and $c_M(\beta)$ such that

$$0 < c_m(\beta) \leqslant \inf_{a,\sigma} c(x, \sigma) \leqslant \sup_{x,\sigma} c(x, \sigma) \leqslant c_M(\beta) < \infty \tag{1.26}$$

Two cases one may want to keep in mind are

$$c(x, \sigma) = \min\{e^{-\beta \Delta_x H(\sigma)}, 1\}$$

and

$$c(x, \sigma) = [1 + \exp(\beta \Delta_x H(\sigma))]^{-1}$$

where

$$\Delta_x H(\sigma) = H^{\sigma}_{\{x\}}(-\sigma(x)) - H^{\sigma}_{\{x\}}(\sigma(x))$$

(H$_1$)–(H$_4$) garantee that there exists a unique Markov process with semigroup $T^{\eta}_V(t)$ and generator L^{η}_V. Here L^{η}_V is a bounded operator on $L^2(\Omega, d\mu^{\eta}_V)$. The process has a unique invariant measure given by μ^{η}_V. Moreover, μ^{η}_V is *reversible* with respect to the process, i.e., L^{η}_V is self-adjoint on $L^2(\Omega, d\mu^{\eta}_V)$. Given $\sigma \in \Omega$, we denote by σ_t the random configuration at time t evolving according to the process, so that

$$\mathbb{E}^{\sigma}_V f(\sigma_t) = \int f(\sigma_t)\, d\mathbb{P}^{\sigma}_V = (T^{\eta}_V(t))\, f(\sigma), \qquad \forall \sigma \in \Omega \quad \text{such that } \sigma_{V^c} = \eta_{V^c}$$

$\mathbb{E}^{\sigma}$ and $\mathbb{P}^{\sigma}$ stand, respectively, for the expectation and the probability measure associated with the process starting from σ_V at time zero and subject to b.c. σ_{V^c}.

The attractivity assumption implies (see, for instance, ref. 10):

1. If f is an increasing function on Ω_V, then $T^{\eta}_V(t)\, f$ is also increasing for all $t \geqslant 0$.
2. If ρ_1, ρ_2 are two probability measures on Ω_V such that $\rho_1 \leqslant \rho_2$, then $\rho_1 T^{\eta}_V(t) \leqslant \rho_2 T^{\eta}_V(t)$ for all $t \geqslant 0$.
3. For any σ, σ' in Ω such that $\sigma \leqslant \sigma'$, the *standard coupling*[10] $\mathbb{P}^{\sigma;\sigma'}_V$ of σ_t, σ'_t is such that $\mathbb{P}^{\sigma;\sigma'}_V\{\sigma_t \leqslant \sigma'_t\} = 1$ for all $t \geqslant 0$.

This last property allows us to define a *standard* coupling of two Gibbs measures which preserve the order of the b.c. Take in fact $\nu^{\eta;\eta'}_V$ as the unique invariant measure of the (standard) coupled process (σ_t, σ'_t). Then we have:

1. $\hat{\nu}_V^{\eta, \eta'}\{(\sigma, \sigma'): \sigma = \sigma_0\} = \mu_V^\eta(\sigma_0)$ for all $\sigma_0 \in \Omega_V$.
2. $\hat{\nu}_V^{\eta, \eta'}\{(\sigma, \sigma'): \sigma' = \sigma_0\} = \mu_V^{\eta'}(\sigma_0)$ for all $\sigma_0 \in \Omega_V$.
3. If $\eta \leqslant \eta'$, then $\hat{\nu}_V^{\eta, \eta'}\{(\sigma, \sigma'): \sigma \leqslant \sigma'\} = 1$.

A fundamental quantity associated with the dynamics of a reversible system is the gap of the generator, i.e.,

$$\text{gap}(V, \eta) = \text{gap}(L_V^\eta) = \inf \text{spec}(-L_V^\eta \upharpoonright \mathbb{1}^\perp)$$

where $\mathbb{1}^\perp$ is the subspace of $L^2(\Omega, d\mu_V^\eta)$ orthogonal to the constant functions. The gap can be also characterized as

$$\text{gap}(V, \eta) = \inf_{f \in L^2(\Omega, d\mu_V^\eta)} \frac{\mathscr{E}_V^\eta(f, f)}{\text{Var}_V^\eta(f)} \tag{1.27}$$

where $\mathscr{E}$ is the Dirichlet form associated with the generator L,

$$\mathscr{E}_V^\eta(f, f) = \tfrac{1}{2} \sum_{\sigma \in \Omega} \sum_{x \in V} \mu_V^\eta(\sigma)\, c(x, \sigma)[f(\sigma^x) - f(\sigma)]^2 \tag{1.28}$$

and Var_V^η is the variance relative to the probability measure μ_V^η.

The main result in this paper is then as follows.

Theorem 1.4. Assume (H_1)–(H_4). If $\beta > \beta_c$, then

$$\lim_{L \to \infty} \left[-\frac{1}{\beta(2L+1)} \log \text{gap}(Q_L, \beta, J = 1, \varnothing) \right] = \tau_\beta$$

2. STRATEGY OF THE PROOF

2.1. Lower Bound

We proceed, following Section 4 of ref. 11, in three steps as follows.

Step 1. We replace the free boundary conditions with very week $+\varepsilon$ b.c., where ε is a small, positive number that will be sent to zero after the thermodynamic limit $L \to \infty$.

Step 2. We prove the sought result for a generalized Glauber dynamics in which single sites are replaced by suitable blocks. This mean that, given *a priori* a covering $\{Q_i\}$ of V_L, at each updating of the dynamics the spin configuration is changed in only one block Q_i and there it is replaced by the equilibrium Gibbs measure of the block given the configuration outside it. It turns out that a convenient choice of the blocks in our case consists of long and thin overlapping rectangles with basis L and height $2\varepsilon L+1$.

Step 3. We relate the gap of the single-site Glauber dynamics to that of the generalized block dynamics in such a way that the estimates obtained in step 2 are not significantly changed when we take the limits $L\to\infty$ and $\varepsilon\to 0$ in the stated order.

More precisely let $J_\varepsilon \equiv J_\varepsilon^\square(Q_L)$ [see (1.6)]. Then, using the variational characterization of the gap in terms of the Dirichlet form (1.27), one gets

$$\operatorname{gap}(Q_L, \beta, 1, \varnothing) \geqslant \frac{c_m}{c_M} e^{-16\beta\varepsilon(2L+1)} \operatorname{gap}(Q_L, \beta, J_\varepsilon, +)$$

where c_m and c_M were defined in (1.26). In order to prove the lower bound on the gap, it is therefore sufficient to show that

$$\lim_{\varepsilon\to 0} \limsup_{L\to\infty} -\frac{1}{\beta(2L+1)} \log[\operatorname{gap}(\beta, Q_L, J_\varepsilon, +)] \leqslant \tau_\beta \tag{2.1}$$

To establish (2.1) we take $M=\lfloor \varepsilon L \rfloor$ and we introduce a covering $\mathscr{R}=\{R_i\}_{i=1}^K$ of Q_L by means of rectangles

$$R_i = Q_{L,M} + (0, h_i)$$

where the h_i are chosen in such a way that the height of the overlap between any two consecutive rectangles $R_i \cap R_{i+1}$ is at least $M/2$ and not greater than M (remember that the height of each R_i is $2M+1$). So we assume that

$$\begin{gathered} h_1 = -L+M, \qquad h_K = L-M \\ M/2 \leqslant h_i - h_{i+1} + (2M+1) = \text{height of the overlap} \leqslant M \end{gathered} \tag{2.2}$$

Let $L_{\mathscr{R}}^{\beta, J_\varepsilon, +}$ be the new generator

$$(L_{\mathscr{R}}^{\beta, J_\varepsilon, +} f)(\sigma) = \sum_{i=1}^K \sum_{\eta\in\Omega_{R_i}} \mu_{R_i}^{\beta, J_\varepsilon, \sigma}(\eta)[f(\sigma^\eta) - f(\sigma)] \tag{2.3}$$

Here, for any $\eta\in\Omega_{R_i}$, σ^η denotes the configuration in Ω_{Q_L} which is equal to η in R_i and to σ in $Q_L\backslash R_i$. The generator $L_{\mathscr{R}}^{\beta, J_\varepsilon, +}$, in the sequel refered to as the block-dynamics generator, is symmetric in $L^2(\Omega_{Q_L}, d\mu_{Q_L}^{\beta, J_\varepsilon, +})$ and we will denote by $\operatorname{gap}(\mathscr{R}, \beta, J_\varepsilon, +)$ the absolute value of its first nonzero eigenvalue.

Theorem 2.1 in ref. 11 relates the gap in the spectrum of the generator of the single-site Glauber dynamics to the gap of generator $L_{\mathscr{R}}^{\beta, J_\varepsilon, +}$ and states that

$$\text{gap}(Q_L, \beta, J_\varepsilon, +) \geqslant \frac{1}{2M(2L+1)} c_m e^{-4\beta(2M+1)} \text{gap}(\mathscr{R}, \beta, J_\varepsilon, +) \quad (2.4)$$

where $c_m = \inf_{\sigma, x} c(x, \sigma)$.

If we now combine (2.4) with the definition of M, we conclude that (2.1) follows once we can prove that

$$\lim_{\varepsilon \to 0} \limsup_{L \to \infty} -\frac{1}{\beta(2L+1)} \log[\text{gap}(\mathscr{R}, \beta, J_\varepsilon, +)] \leqslant \tau_\beta \quad (2.5)$$

Remark. It is important to remark at this point that our choice of the parameters M and ε is different from that of ref. 11. There in fact the only varying parameter was the size L of the square Q_L and the numbers ε, M were directly related to L through the choice

$$\varepsilon = L^{-1/2}; \qquad M = L^{1/2+\delta}$$

where $\delta \ll 1$ was a fixed number. Here, unfortunately, we cannot do that, and we are able to take the limit $\varepsilon \to 0$ only *after* the thermodynamic limit $L \to \infty$. The reason for this difference is that, for β close to the critical value β_c, we are still unable to controls the moderate fluctuations of an interface of length L between the plus and minus phases, i.e., fluctuations that occur on a scale L^α with $\frac{1}{2} < \alpha < 1$ (see, e.g., the proof of Proposition 3.1). At very low temperature the powerful methods of the cluster expansion are available and they give a detailed control of all fluctuations, from the normal ones occurring on scale $\sqrt{L}$ to the large ones on scale L.[2]

The advantage of studying the block dynamics instead of the single-site dynamics is that the relaxation time can be estimated from above via natural probabilistic methods in a rather precise way. Such a probabilistic approach was developed in Section 4 of ref. 11 and its implementation requires only two key equilibrium estimates described in the propositions below. Once these estimates are at hand, then (2.5) follows.

Let us briefly recall the argument of ref. 11 in order explain the different roles played by these propositions.

Let us couple the processes starting from the two extreme configurations which consist of all pluses and all minuses, respectively, in such a way that their order is preserved during the evolution. It is easy to show that the relaxation time of the block dynamics can be estimated from above by, roughly speaking, the inverse of the probability that, after updating in increasing order the blocks $R_1 \cdots R_K$, the two configurations have become identical. In order to estimate from below this probability it is important to distinguish between the first updating of R_1 and the subsequent ones. In the first updating, in fact, the two new configurations will have a strong tendency to disagree in R_1 since the typical configurations of the two Gibbs measures $\mu_{R_1}^{\beta, J_\varepsilon, [0]}$, $\mu_{R_1}^{\beta, J_\varepsilon, +}$ in R_1 have the structure of the minus and plus phases, respectively. However, using Proposition 2.2 below, one can show that the probability of agreement in, e.g., $R_2 \backslash R_1$ is not smaller than $\exp[-\beta(\tau_\beta + 14\varepsilon)(2L+1)]$, for any $\beta > \beta_c$ and any L large enough. On the contrary, once the first updating has forced, via a large deviation, the two configurations to agree in $R_2 \backslash R_1$, then, thanks to the weak $+\varepsilon$ b.c., the next updatings will monotonically enlarge the region of agreement to the next blocks $R_2, R_3, \ldots, R_K$ with probability exponentially in L close to one. The key technical result to establish this second property of the block dynamics is Proposition 2.1 below.

Given these two results, one can conclude that the probability of finding complete agreement at the end of the sequence of updatings is, roughly speaking, of the same order as the probability of agreement in the first updating, i.e., not smaller than $\exp[-\beta(\tau_\beta + 14\varepsilon)(2L+1)]$ and (2.5) follows. We refer the reader to Section 4 of ref. 11 for more details.

Proposition 2.1. Let $\beta > \beta_c$ and let $\varepsilon, \alpha \in (0, 1]$. Given a positive integer L, we set $M = \lfloor \varepsilon L \rfloor$, $k = \lfloor \varepsilon L/10 \rfloor$. Let $\Lambda = Q_{L,M} + (0, h)$ be a vertical translate of $Q_{L,M}$ contained in Q_L and let $\bar{\Lambda}$ be a vertical translate of $Q_{L,N}$ (with $M \leqslant N < L$) such that the bottom sides of Λ and $\bar{\Lambda}$ coincide. We also let

$$\Lambda_{\text{bot}} = \{x = (x_1, x_2) \in \Lambda : x_2 \leqslant M - 3k + h\}$$

Take $J_\alpha = J_\alpha^\square(Q_L)$. Then there exist $L_0 = L_0(\beta, \alpha, \varepsilon)$ and $m = m(\beta, \alpha, \varepsilon)$ such that if $L \geqslant L_0$, we have the following:

(i) If the horizontal sides of Λ do not touch the horizontal sides of Q_L (and so the boundary coupling is identically equal to one on these sides), then

$$\mu_\Lambda^{\beta, J_\alpha, +}\{\sigma(x) = +1\} - \mu_\Lambda^{\beta, J_\alpha, [0]}\{\sigma(x) = +1\} \leqslant e^{-mL} \qquad \forall x \in \Lambda_{\text{bot}} \tag{2.6}$$

(ii) We have

$$\mu_\Lambda^{\beta, J_\alpha, +}\{\sigma(x) = +1\} - \mu_{\bar{\Lambda}}^{\beta, J_\alpha, +}\{\sigma(x) = +1\} \leqslant e^{-mL} \qquad \forall x \in \Lambda_{\text{bot}} \tag{2.7}$$

Remark. The intuitive meaning of the proposition is the following. If we consider a rectangle as above, then the typical configurations of any associated Gibbs measure with full plus b.c. on the bottom side and weak $\varepsilon +$ b.c on the lateral sides will coincide with those of the Gibbs measure with full plus b.c. also on the top side, at least on all the sites not too close to the top side.

Proposition 2.2. Under the same assumptions as in Proposition 2.1, let $\Lambda_{\text{top}} = \Lambda \backslash \Lambda_{\text{bot}}$ and let $\Gamma_\Lambda^{[0]}(\sigma)$ be the unique open contour produced by the configuration σ with boundary condition $[0]$. Consider the event

$$F(L, \varepsilon) = \{\sigma \in \Omega_\Lambda : \Gamma_\Lambda^{[0]}(\sigma) \subset \mathscr{E}^*_{\Lambda_{\text{top}}}\}$$

Then there exists $L_0 = L_0(\beta, \varepsilon)$ such that if $L \geqslant L_0$, then

$$\mu_\Lambda^{\beta, J_\varepsilon, [0]}(F(L, \varepsilon)) \geqslant \exp[-\beta(\tau_\beta + 14\varepsilon)(2L+1)] \tag{2.8}$$

The proof of the propositions is postponed to Section 4.

2.2. Upper Bound

Following ref. 11, let $f(\sigma)$ be the trial function

$$f_\Lambda(\sigma) = \chi\{m_\Lambda(\sigma) > 0\} - \chi\{m_\Lambda(\sigma) < 0\} \tag{2.9}$$

where

$$m_\Lambda(\sigma) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma(x)$$

If we plug f_Λ in the variational characterization of the gap (1.27), we get

$$\text{gap}(Q_L, \beta, \varnothing) \leqslant 4c_M(2L+1)^2 \frac{\mu_{Q_L}^{\beta, \varnothing}\{|m_{Q_L}(\sigma)| \leqslant 2/|Q_L|\}}{1 - \mu_{Q_L}^{\beta, \varnothing}\{m_{Q_L}(\sigma) = 0\}} \tag{2.10}$$

The upper bound then follows once we show that, for any $\beta > \beta_c$,

$$\liminf_{L \to \infty} -\frac{1}{\beta(2L+1)} \log\left[\mu_{Q_L}^{\beta, \varnothing}\left\{|m_{Q_L}(\sigma)| \leqslant \frac{2}{|Q_L|}\right\}\right] \geqslant \tau_\beta \tag{2.11}$$

In the framework of the rigorous Wulff construction, at very low temperature, Shlosman[17] computed the logarithmic asymptotics of the probability of rare events like $\{m_{Q_L}(\sigma) = m\}$ when $m \in (-m^*, m^*)$ (m^* denotes the spontaneous magnetization) and, in particular, he proved (2.11). We are going to show that (2.11) holds at any $\beta \geqslant \beta_c$, i.e., that the following result holds.

Proposition 2.3. Let $\beta > \beta_c$ and $m \in (-m^*, m^*)$. Then

$$\liminf_{L \to \infty} -\frac{1}{\beta(2L+1)} \log[\mu_{Q_L}^{\beta, \varnothing}\{m_{Q_L}(\sigma) = m\}] \geqslant \varphi(m)$$

where

$$\varphi(m) = \frac{1}{2} w \left(\frac{m^*(\beta) - (|m| \vee m_1)}{2m^*(\beta)} \right)^{1/2}$$

where the constant w is the value of the Wulff functional W_τ on the Wulff curve (see, e.g., ref. 17 or ref. 2) and the singularity point m_1 satisfies the equation

$$\frac{1}{2} w \left(\frac{m^*(\beta) - m_1}{2m^*(\beta)} \right)^{1/2} = \tau_\beta$$

The proof of the proposition is postponed to Section 5.

Remark. One can actually strengthen the above proposition by proving that

$$\lim_{\varepsilon \to 0} \lim_{L \to \infty} -\frac{1}{\beta(2L+1)} \log[\mu_{Q_L}^{\beta, \varnothing}\{|m_{Q_L}(\sigma) - m| \leqslant \varepsilon\}] = \varphi(m)$$

We decided, however, to omit the proof of the necessary lower bound since it follows without any significant change the proof given in Section 5 of ref. 7 (see also ref. 20). It is interesting to remark, however, that, at least for all values of m in the interval $[0, m_1]$, the stronger version of the proposition follows at once from the result of the first part of the section. In fact, if in the variational characterization of the gap we replace the trial function $f(\sigma)$ defined in (2.9) with the new function

$$g_{\Lambda, m}(\sigma) = \chi\{m_\Lambda(\sigma) > m\} - \chi\{m_\Lambda(\sigma) < m\}, \qquad m \in [0, m^*)$$

we obtain

$$\frac{\mathrm{Var}_{Q_L}^{\beta, \varnothing}(g_{Q_L, m})}{4c_M(2L+1)^2} \operatorname{gap}(Q_L, \beta, \varnothing) \leqslant \mu_{Q_L}^{\beta, \varnothing} \left\{ |m_{Q_L}(\sigma) - m| \leqslant \frac{2}{|Q_L|} \right\}$$

Since

$$\limsup_{L \to \infty} \mu_{Q_L}^{\beta, \varnothing} \{m_{Q_L}(\sigma) \in [-m^* + \delta, m^* - \delta]\} = 0$$

for any $\delta > 0$ and any $\beta > \beta_c$,[1] one gets that $\lim_{L \to \infty} \mathrm{Var}(g_m) = 1$. Thus, using the correct lower bound on the gap proven in the first part of the section, we conclude that

$$\limsup_{L \to \infty} -\frac{1}{\beta(2L+1)} \log \left[\mu^{\beta, \varnothing}_{Q_L} \; |m_{Q_L}(\sigma) - m| \leqslant \frac{2}{|Q_L|} \right] \leqslant \tau_\beta$$

which is the same as the lower bound proven in the proposition above if $m \in [0, m_1]$.

3. FIRST BASIC RESULTS

We consider the system on a rectangle $Q_{L,M}$ with $[k]$ boundary conditions [see (1.5)]. We let

$$\Gamma^{[k]}_\Lambda(\sigma) = \text{the unique open contour produced by a configuration } \sigma \quad (3.1)$$

The aim of this section is to show the following result.

Proposition 3.1. Let $\beta > \beta_c$ and let $\varepsilon, \alpha \in (0, 1]$. Given a positive integer L, we set $M = \lfloor \varepsilon L \rfloor$, $k = \lfloor \varepsilon L/10 \rfloor$, $\Lambda(L) = Q_{L,M}$, and $J = J^{||}_\alpha(\Lambda)$ [see (1.6)]. We also let

$$V(L) = \{x = (x_1, x_2) \in \Lambda(L): x_2 \geqslant M - 2k + 1\}$$

and consider the event

$$F(L, \varepsilon) = \{\sigma \in \Omega_\Lambda: \Gamma^{[0]}_\Lambda(\sigma) \subset \mathscr{E}^*_{V(L)}\}$$

Then

$$\liminf_{L \to \infty} \left[-\frac{1}{L} \log \mu^{\beta, J, [0]}_{\Lambda(L)}(F(L, \varepsilon)^c) \right] \equiv m(\beta, \alpha, \varepsilon) > 0$$

Proof. Since $F(L, \varepsilon)$ is a positive event, we can use the FKG inequality and write

$$\mu^{\beta, J, [0]}_{\Lambda(L)}(F(L, \varepsilon)^c) \leqslant \mu^{\beta, J, [k]}_{\Lambda(L)}(F(L, \varepsilon)^c)$$

Then we notice that, letting $R = \{x = (x_1, x_2) \in \Lambda^*: x_2 = M - 2k - 1/2\}$, we have

$$F(L, \varepsilon)^c \subset \{\sigma \in \Omega_\Lambda: \Gamma^{[k]}_\Lambda(\sigma) \subset \mathscr{E}^*_{V(L)}\}^c = \bigcup_{x \in R} \{\sigma \in \Omega_\Lambda: \Gamma^{[k]}_\Lambda(\sigma) \ni x\}$$

where by $\Gamma \ni x$ we improperly mean that one edge in Γ has x as one of its endpoints, i.e., that $\mathscr{V}(\Gamma) \ni x$. Using now Lemma 3.2 and Propositions 3.6 and 3.7 given below, one can easily finish the proof. ▌

We start by rewriting the quantity we want to estimate in terms of two-point functions in the dual lattice. Letting, for simplicity,

$$\langle \cdot \rangle^* = \mathbf{E}_{\Lambda^*, \mathscr{E}_\Lambda^*}^{\beta^*; J^*; \varnothing}(\cdot)$$

we have the following result.

Lemma 3.2. Let $\beta > 0$, $J \geqslant 0$, let L, M be two positive integers, and let $\Lambda = Q_{L,M}$. Choose now an integer k with $0 \leqslant k \leqslant 2M+1$. Let a and b be the two extreme sites in the $(k+1)$th row (from the top) in Λ^*, i.e.,

$$a = (-L - \tfrac{1}{2}, M + \tfrac{1}{2} - k), \qquad b = (+L + \tfrac{1}{2}, M + \tfrac{1}{2} - k) \tag{3.2}$$

Then for each $x \in \Lambda^*$ we have

$$\mu_\Lambda^{\beta, J, [k]}\{\sigma \in \Omega_\Lambda : \Gamma_\Lambda^{[k]}(\sigma) \ni x\} \leqslant \frac{\langle \sigma(a)\, \sigma(x) \rangle^* \, \langle \sigma(x)\, \sigma(b) \rangle^*}{\langle \sigma(a)\, \sigma(b) \rangle^*} \tag{3.3}$$

Proof. If x coincides with a or b, the statement is trivial, so we assume $x \neq a, b$. For all $\sigma \in \Omega_\Lambda$, we let $\Gamma = \Gamma_\Lambda^{[k]}(\sigma)$. If $\Gamma \ni x$, then it is possible to represent Γ as

$$\Gamma = (e_1, \dots, e_l, e_{l+1}, \dots, e_m)$$

in such a way that $a \in \delta e_1$, $b \in \delta e_m$, and e_l is the first edge whose boundary contains x. We then let

$$\Gamma_1 = (e_1, \dots, e_l) \qquad \text{and} \qquad \Gamma_2 = (e_{l+1}, \dots, e_m)$$

Γ_1 is thus an open contour in $\mathscr{E}_\Lambda^*$ with boundary $\{a, x\}$, but it has the additional property that there is exactly one edge e in Γ_1 such that $\delta e \ni x$, i.e., Γ_1 cannot come back to x after a first visit. We denote by $C(\mathscr{E}_\Lambda^*, \{a, \underline{x}\})$ the set of all open contours which satisfy this further requirement.

As far as Γ_2 is concerned, we observe that, since Γ is a contour, then none of the edges of Γ_2 with the exception of perhaps the first lies in $\Delta(\Gamma_1)$. Let then $\bar{e}$ be the unique forbidden edge of e_l (not equal to e_l) which originates from x. Then it is possible that $e_{l+1} = \bar{e}$ unless $\bar{e}$ is forbidden by another edge in Γ_1. For this reason we claim that the set of all possible pairs (Γ_1, Γ_2) is equal to the set [recall (1.14)]

$$\{(\Gamma', \Gamma''): \Gamma' \in C(\mathscr{E}_\Lambda^*, \{a, \underline{x}\}), \Gamma'' \in C(\mathscr{E}_\Lambda^* \backslash \Delta'(\Gamma', x), \{x, b\})\}$$

Then we can write the "numerator" of the LHS of (3.3) as

$$Z^{\beta, J, [k]}(\Lambda)\, \mu_\Lambda^{\beta, J, [k]}\{\Gamma \ni x\}$$

$$= \sum_{\substack{\Gamma \in C(\mathscr{E}_\Lambda^*, \{a, b\}) \\ \Gamma \ni x}} w(\Gamma)\, \tilde{Z}(\mathscr{E}_\Lambda^* \backslash \Delta(\Gamma))$$

$$= \sum_{\Gamma_1 \in C(\mathscr{E}_\Lambda^*, \{a, \underline{x}\})} w(\Gamma_1) \sum_{\Gamma_2 \in C(\mathscr{E}_\Lambda^* \backslash \Delta'(\Gamma_1, x), \{x, b\})} w(\Gamma_2)\, \tilde{Z}(\mathscr{E}_\Lambda^* \backslash \Delta(\Gamma)) \tag{3.4}$$

In general $\Delta'(\Gamma_1, x) \cup \Delta(\Gamma_2) \neq \Delta(\Gamma_1) \cup \Delta(\Gamma_2) = \Delta(\Gamma)$; nevertheless, if γ is a *closed* contour which does not intersect $\Delta'(\Gamma_1, x) \cup \Delta(\Gamma_2)$, then one can check, by considering all cases, that γ does not contain the possible extra edge $\bar{e}$ forbidden by Γ. So we have

$$\tilde{Z}(\mathscr{E}_\Lambda^* \backslash \Delta(\Gamma)) = \tilde{Z}(\mathscr{E}_\Lambda^* \backslash [\Delta'(\Gamma_1, x) \cup \Delta(\Gamma_2)])$$

As a consequence, the product of the last two terms in (3.4) can be written as

$$\tilde{Z}^{\beta, J}(\mathscr{E}_\Lambda^* \backslash \Delta'(\Gamma_1, x), \{x, b\})$$

which, by (1.21), becomes

$$\mathbf{E}^{\beta^*, J^*, \varnothing}_{\Lambda^*, \mathscr{E}_\Lambda^* \backslash \Delta'(\Gamma_1, x)}(\sigma(x)\, \sigma(b))\, \tilde{Z}(\mathscr{E}_\Lambda^* \backslash \Delta'(\Gamma_1, x))$$

Using the second Griffiths inequality, we obtain

$$\text{LHS of (3.4)} \leqslant \langle \sigma(x)\, \sigma(b) \rangle^* \sum_{\Gamma_1 \in C(\mathscr{E}_\Lambda^*, \{a, \underline{x}\})} w(\Gamma_1)\, \tilde{Z}(\mathscr{E}_\Lambda^* \backslash \Delta'(\Gamma_1, x)) \tag{3.5}$$

A straightforward check shows now that the simultaneous substitution

$$C(\mathscr{E}_\Lambda^*, \{a, \underline{x}\}) \to C(\mathscr{E}_\Lambda^*, \{a, x\})$$

$$\Delta'(\Gamma_1, x) \to \Delta(\Gamma_1)$$

has no effect one the RHS of (3.5). Thus, using (1.21) again, we get

$$\mu_\Lambda^{\beta, J, [k]}\{\Gamma \ni x\} \leqslant \frac{\tilde{Z}(\mathscr{E}_\Lambda^*)}{Z^{\beta, J, [k]}(\Lambda)} \langle \sigma(a)\, \sigma(x) \rangle^* \langle \sigma(x)\, \sigma(b) \rangle^*$$

which is equivalent to (3.3). ∎

The next proposition is a fairly general inequality which will be used later. We use ideas contained in ref. 14.

Proposition 3.3. Let $\Lambda \subset\subset \mathbb{Z}^2$, $x \in \Lambda$, and $y \in \partial^+ \Lambda$. Assume there is a family of nested subsets

$$\Lambda = \Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_n \ni x$$

For each $i = 1,..., n$, let $Y_i = \partial^+ \Lambda_i \cap (\Lambda \cup \{y\})$, and let U_i be an arbitrary subset of Y_i. Let $|J|_\infty = \sup_{u, v \in \mathbb{Z}^2} J(u, v)$. For each $\tau \in \Omega$ we define the boundary conditions

$$(\tau_\pm)(u) = \begin{cases} \tau(u) & \text{if } u \in (\Lambda \cup \{y\})^c \\ \pm 1 & \text{if } u \in \Lambda \cup \{y\} \end{cases} \tag{3.6}$$

Then we have

$$[\mu_\Lambda^{\beta, J, \tau_+}\{\sigma(x) = 1\} - \mu_\Lambda^{\beta, J, \tau_-}\{\sigma(x) = 1\}] \leqslant \prod_{i=0}^{n-1} d_i \tag{3.7}$$

where

$$d_i = 1 - (\tfrac{1}{2} e^{-8\beta |J|_\infty})^{2|U_{i+1}|}$$
$$\times \left[1 - \sum_{u \in Y_{i+1} \setminus U_{i+1}} [\mu_{\Lambda_i}^{\beta, J, \tau_+}\{\sigma(u) = 1\} - \mu_{\Lambda_i}^{\beta, J, \tau_-}\{\sigma(u) = 1\}]\right]$$

Proof. In this proof β and J are fixed and we usually omit them. Let

$$a_i = \mu_{\Lambda_i}^{\tau_+}\{\sigma(x) = 1\} - \mu_{\Lambda_i}^{\tau_-}\{\sigma(x) = 1\}$$

[notice that in particular a_0 is the LHS of (3.7)]. Then we claim that

$$a_i \leqslant b_i a_{i+1} \tag{3.8}$$

where

$$b_i = \inf_{\nu \in K, \mu_{\Lambda_i}^{\tau_-}} b_i(\nu)$$

$$b_i(\nu) = \nu\{\eta \neq \eta' \text{ on } Y_{i+1}\} \equiv \nu\{\exists z \in Y_{i+1} \text{ such that } \eta(z) \neq \eta'(z)\}$$

and $K(\mu_1, \mu_2)$ denotes the set of all couplings (joint representations) of μ_1 and μ_2. In fact, by the DLR property, for each coupling $\nu \in K(\mu_{\Lambda_i}^{\tau_+}, \mu_{\Lambda_i}^{\tau_-})$,

$$a_i = \sum_{\eta, \eta' \in \Omega} \nu(\eta, \eta')[\mu_{\Lambda_{i+1}}^{\eta}\{\sigma(x) = 1\} - \mu_{\Lambda_{i+1}}^{\eta'}\{\sigma(x) = 1\}]$$

Since the sum is actually restricted over those η, η' which agree with τ outside Λ, the quantity inside the brackets vanishes unless $\eta \neq \eta'$ on Y_{i+1} and

(3.8) follows. So, what is left is to show that there is a coupling $\bar{\nu}$ such that $b_i(\bar{\nu}) \leqslant d_i$. To do that we apply the so-called "surgery technique"[3] to the standard coupling $\hat{\nu}$ as in ref. 14, i.e., we define for each $\tau, \tau' \in \Omega$

$$\bar{\nu}_{\Lambda_i}^{\tau, \tau'}(\sigma, \sigma') = \sum_{\eta, \eta'} \hat{\nu}_{\Lambda_i}^{\tau, \tau'}(\eta, \eta')\, \mu_{U_{i+1}}^{\eta}(\sigma)\, \mu_{U_{i+1}}^{\eta'}(\sigma')$$

It can be shown (see, for instance, ref. 14) that $\bar{\nu}_{\Lambda_i}^{\tau, \tau'} \in K(\mu_{\Lambda_i}^{\tau}, \mu_{\Lambda_i}^{\tau'})$. Then, letting $W_i = Y_i \backslash U_i$, we get

$$\begin{aligned} b_i(\bar{\nu}_{\Lambda_i}^{\tau_+, \tau_-}) &= 1 - \bar{\nu}_{\Lambda_i}^{\tau_+, \tau_-}\{\sigma = \sigma' \text{ on } Y_{i+1}\} \\ &\leqslant 1 - \min_{\eta \in \Omega} (\mu_{U_{i+1}}^{\eta}\{\sigma(x) = 1\ \forall x \in U_{i+1}\})^2\, \hat{\nu}_{\Lambda_i}^{\tau_+, \tau_-}\{\sigma = \sigma' \text{ on } W_{i+1}\} \end{aligned}$$

Hence, by FKG, we have

$$\begin{aligned} \mu_{U_{i+1}}^{\eta}\{\sigma(x) = 1\ \forall x \in U_{i+1}\} &\geqslant \prod_{x \in U_{i+1}} \mu_{U_{i+1}}^{\eta}\{\sigma(x) = 1\} \\ &\geqslant [\tfrac{1}{2} e^{-8\beta |J|_\infty}]^{|U_{i+1}|} \end{aligned} \tag{3.9}$$

On the other hand,

$$\begin{aligned} &\hat{\nu}_{\Lambda_i}^{\tau_+, \tau_-}\{\sigma = \sigma' \text{ on } W_{i+1}\} \\ &\quad \geqslant 1 - \sum_{x \in W_{i+1}} \hat{\nu}_{\Lambda_i}^{\tau_+, \tau_-}\{\sigma(x) \neq \sigma'(x)\} \\ &\quad = 1 - \sum_{x \in W_{i+1}} [\mu_{\Lambda_i}^{\tau_+}\{\sigma(x) = 1\} - \mu_{\Lambda_i}^{\tau_-}\{\sigma(x) = 1\}] \end{aligned}$$

which, together with (3.9), implies $b_i(\bar{\nu}_{\Lambda_i}^{\tau_+, \tau_-}) \leqslant d_i$. ∎

We now use previous proposition to show the following result.

Proposition 3.4. For each $\beta < \beta_c$ and $\alpha > 0$ there exist $m_2(\beta, \alpha) > 0$ and $q_2(\beta, \alpha)$ such that, if $\Lambda = Q_{L, M}$ and J is a set of couplings satisfying (i) $\sup_{x, y} J(x, y) \equiv |J|_\infty \leqslant \alpha$ and (ii) $J(x, y) = 1$ if $\{x, y\} \cap (\Lambda \backslash \partial \Lambda) \neq 0$, then for all $\tau \in \Omega$ and for all $x, y \in \Lambda$ such that $|x - y| \geqslant q_2(\beta, \alpha)$, we have

$$\mathrm{Cov}_{\Lambda}^{\beta, J, \tau}(\sigma(x), \sigma(y)) \leqslant \exp(-m_2 \sqrt{|x - y|}) \tag{3.10}$$

$$\mathbf{E}_{\Lambda}^{\beta, J, \varnothing}\, \sigma(x)\, \sigma(y) \leqslant \exp(-m_2 \sqrt{|x - y|}) \tag{3.11}$$

Proof. Choose x and y in Λ and $\tau \in \Omega$. Let then

$$\tau_{\pm}(u) = \begin{cases} \tau(u) & \text{if } u \in \Lambda^c \\ \pm 1 & \text{if } u \in \Lambda \end{cases}$$

Let also $\Lambda_y = \Lambda \backslash \{y\}$. A straightforward computation shows that

$$\mathrm{Cov}_\Lambda^{\beta, J, \tau}(\sigma(x), \sigma(y)) = 4D\mu_\Lambda^{\beta, J, \tau}\{\sigma(y) = +1\}\ \mu_\Lambda^{\beta, J, \tau}\{\sigma(y) = -1\}$$

where

$$D = \mu_{\Lambda_y}^{\beta, J, \tau+}\{\sigma(x) = +1\} - \mu_{\Lambda_y}^{\beta, J, \tau-}\{\sigma(x) = +1\} \tag{3.12}$$

Let now $s = |x - y|$ and $k = (\lfloor \sqrt{2} \rfloor - 10) \vee 0$. It is possible to find k positive integers $l_1, ..., l_k$ such that:

(a) $l_i \geqslant l_{i+1} + k$ and $s \geqslant l_1 + k$.

(b) If we set $\Lambda_0 = \Lambda_y$ and, for each $i = 1, ..., k$, $\Lambda_i \equiv \Lambda \cap Q_{l_i}(x)$ and $Y_i = \partial^+ \Lambda_i \cap \Lambda$, then we have

$$\#\{u \in Y_{i+1} : d(u, \partial\Lambda_i) = j\} \leqslant 2 \qquad \forall j \leqslant k - 2 \tag{3.13}$$

For each $i = 1, ..., k$ and $u \in \Lambda_i$ we define $B_i(u)$ as the largest square centered on u, contained in Λ_i and such that $J(v, z) = 1$ if either v or z is in $B_i(u)$. We have then

$$B_i(u) = Q_{\alpha_i(u)}(u)$$

where

$$\alpha_i(u) = d(u, \partial\Lambda_i) - 1$$

Because of the definition of the $B_i(u)$ we have

$$\mu_{\Lambda_i}^{\beta, J, \tau+}\{\sigma(u) = 1\} - \mu_{\Lambda_i}^{\beta, J, \tau-}\{\sigma(u) = 1\}$$
$$\leqslant \mu_{B_i(u)}^{\beta, J = 1, +}\{\sigma(u) = 1\} - \mu_{B_i(u)}^{\beta, J = 1, -}\{\sigma(u) = 1\}$$

Thanks to Theorem 2 in ref. 5 we know that there exist $C(\beta)$ and $m_1(\beta) > 0$ such that, for each $u \in \Lambda_i$,

$$\mu_{B_i(u)}^{\beta, +}\{\sigma(u) = 1\} - \mu_{B_i(u)}^{\beta, -}\{\sigma(u) = 1\} \leqslant C(\beta)\, e^{-m_1(\beta)\, \alpha_i(u)} \tag{3.14}$$

We take an integer $r(\beta)$ such that

$$C(\beta) \sum_{j = r(\beta)}^{\infty} e^{-m_1(\beta) j} \leqslant \tfrac{1}{8} \tag{3.15}$$

and let

$$U_i = \{u \in Y_i : d(u, \partial\Lambda) \leqslant r(\beta)\}$$

If we take L large enough, we can assume $k \geqslant r+2$ and, thanks to (3.13),

$$|U_i| \leqslant 2(r+1) \tag{3.16}$$

In this way, letting $W_i = Y_i \backslash U_i$ and using (3.14), we obtain

$$A_i \equiv \sum_{u \in W_{i+1}} [\mu_{A_i}^{\beta, J, \tau+}\{\sigma(u)=1\} - \mu_{A_i}^{\beta, J, \tau-}\{\sigma(u)=1\}]$$
$$\leqslant \sum_{j=0}^{\infty} C(\beta)\, e^{-m_1(\beta) j} \# \{u \in W_{i+1} : \alpha_i(u) = j\}$$

By (3.13) and the trivial bound $|W_i| \leqslant 4(2l_i+1)$, we get

$$A_i \leqslant \sum_{j=r}^{k-2} 2C(\beta)\, e^{-m_1(\beta) j} + 4(2l_{i+1}+1)\, C(\beta)\, e^{-m_1(\beta)(k-1)}$$
$$\leqslant 2\tfrac{1}{8} + 9sC(\beta)\, e^{-m_1(\beta)(k-1)}$$

which implies

$$A_i \leqslant \tfrac{1}{2} \tag{3.17}$$

if $s = |x-y| \geqslant q_2(\beta, \alpha)$ with $q_2(\beta, \alpha)$ large enough.

By (3.12), Proposition 3.3, (3.16) and (3.17), we find

$$D \leqslant [1 - \tfrac{1}{2}(\tfrac{1}{2} e^{-8\beta |J|_\infty})^{2r+1}]^k$$

which gives (3.10). Inequality (3.11) is obtained by taking $J(u, v) = 0$ if $[u, v]^* \in \delta\Lambda$. ∎

The previous result can be improved to get an exponential decay, thanks to Simon's inequality.

Corollary 3.5. For each $\beta < \beta_c$ and $\alpha > 0$ there exist $m_3(\beta, \alpha) > 0$ and $C_3(\beta, \alpha)$ such that, if Λ and J are as in Proposition 3.4, then for all $x, y \in \Lambda$ we have

$$\mathbf{E}_\Lambda^{\beta, J, \varnothing} \sigma(x)\, \sigma(y) \leqslant C_3 e^{-m_3 |x-y|} \tag{3.18}$$

Proof. Let $l_0 = l_0(\beta, \alpha) > q_2(\beta, \alpha)$ be such that

$$4(2l_0+1)\, e^{-m_2 \sqrt{l_0}} \equiv b < 1$$

and let

$$f(u, v) = \mathbf{E}_\Lambda^{\beta, J, \varnothing} \sigma(u)\, \sigma(v), \qquad u, v \in \Lambda$$

Theorem 2.1 in ref. 18 says that, if $|x-y|>l_0$, then

$$f(x, y) \leqslant \sum_{z \in \Lambda: |z-x|=l_0} f(x, z) f(z, y)$$

Iterating this inequality, we get

$$f(x, y) \leqslant b^{\lfloor l_0^{-1} |x-y| \rfloor}$$

which proves the corollary. ▌

Proposition 3.6. For each $\beta<\beta_c, \alpha>0$, and $t>0$ there exist $C_4(\beta, \alpha, t)$ and $m_4(\beta, \alpha, t)>0$ such that, if $\Lambda=Q_{L,M}$ and J is a set of couplings satisfying (i) $\sup_{x,y} J(x, y) \equiv |J|_\alpha \leqslant \alpha$ and (ii) $J(x, y)=1$ unless [let $x=(x_1, x_2)$, $y=(y_1, y_2)$] $x_1=y_1=-L$ or $x_1=y_1=+L$, then for all $x, y \in \Lambda$ such that $|x_2-y_2| \geqslant t\,|x_1-y_1|$ we have

$$\mathbf{E}_\Lambda^{\beta, J, \varnothing} \sigma(x)\, \sigma(y) \leqslant C_4 \exp(-\beta^* \tau_{\beta^*} |x_1-y_1| - m_4 |x_2-y_2|) \qquad (3.19)$$

where τ_{β^*} is the surface tension at zero degrees.

Proof. The idea is to use the Lieb improvement of Simon's inequality in order to get rid of those sites where $J \neq 1$ and obtain the surface tension in this way.

Define

$$f_\Lambda(u, v)=\mathbf{E}_\Lambda^{\beta, J, \varnothing} \sigma(u)\, \sigma(v), \qquad u, v \in \Lambda$$

If $|x_1-y_1|<3$, then (3.19) is a consequence of (3.18). Assume then $y_1 \geqslant x_1+3$ and let

$$B_1=\{z=(z_1, z_2) \in \Lambda: z_1=x_1+1\}$$

$$B_2=\{z=(z_1, z_2) \in \Lambda: z_1=y_1-1\}$$

Define also

$$\Lambda'=\{z=(z_1, z_2) \in \Lambda: z_1 \geqslant x_1+1\}$$

$$\Lambda''=\{z=(z_1, z_2) \in \Lambda: x_1+1 \leqslant z_1 \leqslant y_1-1\}$$

By Lieb's improvement[10] of Simon's inequality (as given in Theorem 2.2 in ref. 18) and by Griffiths' second inequality we get

$$f_\Lambda(x, y) \leqslant \sum_{z \in B_1} f_\Lambda(x, z) f_{\Lambda'}(z, y) \leqslant \sum_{z \in B_1} \sum_{u \in B_2} f_\Lambda(x, z) f_{\Lambda''}(z, u) f_\Lambda(u, y)$$

Since $J=1$ on Λ'' we can write [using (1.10) and (3.18)]

$$f_\Lambda(x, y) \leqslant C_3(\beta, \alpha)^2 \sum_{z \in B_1} \sum_{u \in B_2} \exp[-\beta^* \tau_{\beta^*}(z-u) - m_3(\beta, \alpha)(|x-z| + |u-y|)] \tag{3.20}$$

Let $0 \leqslant \vartheta \leqslant \pi/2$ be such that

$$|z_2 - u_2| = |z_1 - u_1| \tan \vartheta$$

The RHS of (3.20) can be written as $A_1 + A_2$, where A_1 is the contribution of those terms with

$$\sin(\vartheta) \geqslant s \equiv \frac{t \wedge 1}{100}$$

and A_2 is the sum of the other terms. Using (1.11), we get for each term in A_1

$$\begin{aligned}
&\beta^* \tau_{\beta^*}(z-u) + m_3(\beta, \alpha)(|x-z| + |u-y|) \\
&\quad \geqslant \beta^* \tau_{\beta^*}(0)\, |z_1 - u_1| + \beta^* D s^2\, |z-u|_2 + \beta m_3(\beta, \alpha)(|x-z| + |u-y|) \\
&\quad \geqslant \beta^* \tau_{\beta^*}\, |x_1 - y_1 - 2| + \left(\beta^* D s^2 \wedge \beta \frac{m_3}{2}\right) |x_2 - y_2| \\
&\qquad + \beta \frac{m_3}{2}(|x-z| + |u-y|)
\end{aligned}$$

while for the terms in A_2 (remember that $|x_2 - y_2| \geqslant t\,|x_1 - y_1|$) we write

$$\begin{aligned}
&\beta^* \tau_{\beta^*}(z-u) + m_3(\beta, \alpha)(|x-z| + |u-y|) \\
&\quad \geqslant \beta^* \tau_{\beta^*}(0)\, |z_1 - u_1| + \beta \frac{m_3}{2}(|x_2 - y_2| - |u_2 - z_2|) \\
&\qquad + \beta \frac{m_3}{2}(|x-z| + |u-y|) \\
&\quad \geqslant \beta^* \tau_{\beta^*}\, |x_1 - y_1 - 2| + \beta \frac{m_3}{4} |x_2 - y_2| + \beta \frac{m_3}{2}(|x-z| + |u-y|)
\end{aligned}$$

In this way we obtain

$$A_1 + A_2 \leqslant C_4 \exp[-\beta^* \tau_{\beta^*}(0)\, |x_1 - y_1 - 2| - \beta m_4\, |x_2 - y_2|] \quad \blacksquare$$

Proposition 3.7. Under the same hypotheses as in Proposition 3.1, we have

$$\liminf_{L\to\infty}\left[-\frac{1}{(2L+1)}\log \mathbf{E}_{\Lambda^*}^{\beta^*, J^*, \varnothing}\sigma(a)\,\sigma(b)\right]\leqslant \beta\tau_\beta$$

Proof. We just need to observe that, thanks to the second Griffiths inequality, since $J^*\geqslant 1$,

$$\mathbf{E}_{\Lambda^*}^{\beta^*, J^*, \varnothing}\sigma(a)\,\sigma(b)\geqslant \mathbf{E}_{V(L)^*}^{\beta^*, \varnothing}\sigma(a)\,\sigma(b)$$

which, by duality, is equal to

$$\frac{Z_V^{\beta, [k]}(L)}{Z_V^{\beta, +}(L)}$$

The proposition then follows from the definition of surface tension. ∎

4. PROOF OF PROPOSITIONS 2.1 AND 2.2

4.1. Proof of Proposition 2.1

To prove part (i), let V be the top portion of Λ of height $2k$, i.e.,

$$V=\{x=(x_1, x_2)\in\Lambda: x_2\geqslant h+M-2k+1\}$$

and let $F(L, \varepsilon)$ be the event

$$F(L, \varepsilon)=\{\sigma\in\Omega_\Lambda: \Gamma_\Lambda^{[0]}(\sigma)\subset\mathscr{E}_V^*\}$$

Then, using FKG, one shows (see the appendix in ref. 11) that for any $x\in\Lambda_{\text{bot}}$

$$\mu_\Lambda^{\beta, J, +}\{\sigma(x)=+1\}-\mu_\Lambda^{\beta, J, [0]}\{\sigma(x)=+1\}\leqslant\mu_\Lambda^{\beta, J, [0]}(F(L, \varepsilon)^c) \tag{4.1}$$

which, combined with Proposition 3.1, yields (i) of Proposition 2.1.

To prove the second inequality we observe that, using the DLR equation and a standard coupling argument, it is sufficient to prove the required bound only for $x=(x_1, x_2)$ with $x_2=h+M-3k$. We set

$$\begin{bmatrix}\Lambda & J & \eta\\ \Lambda' & J' & \eta'\end{bmatrix}=\mu_\Lambda^{\beta, J, \eta}\{\sigma(x)=+1\}-\mu_{\Lambda'}^{\beta, J', \eta'}\{\sigma(x)=+1\}$$

and, for simplicity, we denote by $\square$ the coupling $J_a^\square$ and so on [see (1.6)]. There are three cases to consider.

Case 1. $\delta_b \Lambda = \delta_b \bar{\Lambda} = \delta_b Q_L$. The LHS of (2.7) can be written as

$$\begin{bmatrix} \Lambda \sqcup + \\ \bar{\Lambda} \sqcup + \end{bmatrix} \leqslant \begin{bmatrix} \Lambda \parallel + \\ \bar{\Lambda} \sqcup + \end{bmatrix} = \begin{bmatrix} \Lambda \parallel + \\ \Lambda \parallel [0] \end{bmatrix} + \begin{bmatrix} \Lambda \parallel [0] \\ \bar{\Lambda} \parallel + \end{bmatrix} + \begin{bmatrix} \bar{\Lambda} \parallel + \\ \bar{\Lambda} \sqcup + \end{bmatrix} \equiv A + B + C$$

A and C can be estimated with (2.6) (for C one has to flip $\bar{\Lambda}$ upside down), while B is nonpositive.

Case 2. $\delta_b \Lambda = \delta_b \bar{\Lambda} \neq \delta_b Q_L$ and $\delta_t \bar{\Lambda} \neq \delta_t Q_L$. In this case we write

$$\text{LHS of (2.7)} = \begin{bmatrix} \Lambda \parallel + \\ \bar{\Lambda} \parallel + \end{bmatrix} = A + B$$

Case 3. $\delta_b \Lambda = \delta_b \bar{\Lambda} \neq \delta_b Q_L$ and $\delta_t = \delta_t Q_L$. We have

$$\text{LHS of (2.7)} = \begin{bmatrix} \Lambda \parallel + \\ \bar{\Lambda} \sqcap + \end{bmatrix} = \begin{bmatrix} \Lambda \parallel + \\ \bar{\Lambda} \parallel [0] \end{bmatrix} + \begin{bmatrix} \Lambda \parallel [0] \\ \bar{\Lambda} \parallel + \end{bmatrix} \leqslant \begin{bmatrix} \Lambda \parallel + \\ \Lambda \parallel [0] \end{bmatrix} \equiv A \quad \blacksquare$$

4.2. Proof of Proposition 2.2

Case 1. $\delta_b \Lambda \neq \delta_b Q_L$. In this case, by monotonicity,

$$\text{LHS of (2.8)} \geqslant \mu_\Lambda(F(L, \varepsilon); \beta, J_\varepsilon^{\parallel}(\Lambda), [0]) \tag{4.2}$$

(the inequality is strict only if $\delta_t \Lambda = \delta_t Q_L$). The RHS of (4.2) tends to one as $L \to \infty$ by Proposition 3.1.

Case 2. $\delta_b \Lambda = \delta_b Q_L$. We first observe that

$$\mu_\Lambda^{\beta, J_\varepsilon, [0]}(F(L, \varepsilon)) \geqslant e^{-8\beta\varepsilon(2L+1)} \mu_\Lambda^{\beta, \bar{J}_\varepsilon, [0]}(F(L, \varepsilon))$$

where

$$\bar{J}_\varepsilon(x, y) = \begin{cases} \varepsilon & \text{on the bottom side of } \Lambda \\ 1 & \text{elsewhere} \end{cases}$$

Define now a, b as in (3.2) with $k = 0$. Let also denote with by A and B the two subsets of $\Lambda \backslash \Delta^s(\Gamma)$ that are respectively above and below Γ. Then we can write

$$\mu_\Lambda^{\beta, \bar{J}_\varepsilon, [0]}(F(L, \varepsilon)) = \sum_{\Gamma \in C(\mathscr{S}^*_{\text{top}}, \{a, b\})} w_{\bar{J}_\varepsilon}(\Gamma) \frac{Z^{\beta, -}(A)\, Z^{\beta, \bar{J}_\varepsilon, +}(B)}{Z^{\beta, \bar{J}_\varepsilon, [0]}(\Lambda)} \tag{4.3}$$

Notice that for any $\Gamma \subset \mathscr{E}^*_{\Lambda_{\rm top}}$, we have $w_{\bar{J}_\varepsilon}(\Gamma) = w(\Gamma)$. Using now the spin-flip symmetry, we obtain

$$\begin{aligned} Z^{\beta, \bar{J}_\varepsilon, [0]}(\Lambda) &\leqslant \exp[2\beta(|\delta_e \Lambda| + |\delta_w \Lambda|) + 2\beta\varepsilon\, |\delta_b \Lambda|]\, Z^{\beta, \bar{J}_\varepsilon, -}(\Lambda) \\ &\leqslant [\exp(9\beta\varepsilon L)]\, Z^{\beta, \bar{J}_\varepsilon, +}(\Lambda) \end{aligned} \tag{4.4}$$

Next we consider the ratio

$$G(\varepsilon) \equiv \frac{Z^{\beta, \bar{J}_\varepsilon, +}(B)}{Z^{\beta, \bar{J}_\varepsilon, +}(\Lambda)}$$

and we claim that there exists $m = m(\beta, e) > 0$ such that

$$\frac{Z^{\beta, \bar{J}_\varepsilon, +}(B)}{Z^{\beta, \bar{J}_\varepsilon, +}(\Lambda)} \geqslant \frac{Z^{\beta, +}(B)}{Z^{\beta, +}(\Lambda)} \exp(-\beta e^{-mL}) \tag{4.5}$$

To prove it we compute the logarithmic derivative of G, obtaining

$$\begin{aligned} \frac{d}{da} \log G(a) &= \beta \sum_{\substack{x = (x_1, x_2) \in \partial\Lambda \\ x_2 = -M}} [\mathbf{E}_B^{\beta, \bar{J}_a, +}(\sigma(x)) - \mathbf{E}_\Lambda^{\beta, \bar{J}_a, +}(\sigma(x))] \\ &\leqslant \beta \sum_{\substack{x = (x_1, x_2) \in \partial\Lambda \\ x_2 = -M}} [\mathbf{E}_{\Lambda_{\rm bot}}^{\beta, \bar{J}_a, +}(\sigma(x)) - \mathbf{E}_\Lambda^{\beta, \bar{J}_a, +}(\sigma(x))] \end{aligned} \tag{4.6}$$

Part (ii) of Proposition 2.1 applied to $\Lambda_{\rm bot}$ and Λ implies that

$$\text{RHS of } (4.6) \leqslant \beta(2L+1)\, 2e^{-m(\beta, a, \varepsilon')L}$$

where m is the quantity given in Proposition 2.1 and ε' is such that the height of $\Lambda_{\rm bot}$ is given by $\lfloor \varepsilon' L \rfloor$. If we redefine

$$m(\beta, \varepsilon) = \tfrac{1}{2} \inf_{a \in [\varepsilon', 1]} m(\beta, a, \varepsilon')$$

we get (4.5) for L large enough.

If we now combine (4.4), (4.5) and use Proposition 3.1, we find

$$\begin{aligned} \text{RHS of } (4.3) &\geqslant \exp(-9\beta\varepsilon L - \beta e^{-mL}) \frac{Z^{\beta, [0]}(\Lambda)}{Z^{\beta, +}(\Lambda)} \mu_\Lambda^{\beta, [0]}(F(L, \varepsilon)) \\ &\geqslant \exp(-10\beta\varepsilon L) \frac{Z^{\beta, [0]}(\Lambda)}{Z^{\beta, +}(\Lambda)} \end{aligned} \tag{4.7}$$

On the other hand, we have

$$\frac{Z^{\beta,[0]}(\Lambda)}{Z^{\beta,+}(\Lambda)} \geqslant e^{-\varepsilon\beta L} \frac{Z^{\beta,[k]}(\Lambda)}{Z^{\beta,+}(\Lambda)} \tag{4.8}$$

Finally, we use duality together with Proposition 3.7 to get that

$$\limsup_{L\to\infty} -\frac{1}{\beta(2L+1)} \log\left[\frac{Z^{\beta,[k]}(\Lambda)}{Z^{\beta,+}(\Lambda)}\right] \leqslant \tau_\beta$$

Thus we can conclude that there exists $L_0(\beta, \varepsilon)$ such that, if $L \geqslant L_0$, then we have

$$\begin{aligned}\mu_\Lambda^{\beta, J_\varepsilon, [0]}(F(L, \varepsilon)) &\geqslant e^{-8\beta\varepsilon(2L+1) - 11\beta\varepsilon L} e^{-\beta\tau_\beta(2L+1)} \\ &\geqslant e^{-\beta(\tau_\beta + 14\varepsilon)(2L+1)} \quad \blacksquare\end{aligned}$$

5. UPPER BOUND ON THE PROBABILITY OF A LARGE DEVIATION

5.1. The Variational Problem

We start by stating in a precise way the variational problem which is preliminary to the proof of Proposition 2.3.

We denote by $\mathscr{D}$ the set of all rectifiable curves $\gamma \in \mathbb{R}^2$ such that γ is either a closed curve inside the unit open square $Q = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ or it is an open curve, which, with the exception of its end-points, is entirely contained in Q. We let $\mathscr{D}_s$ denote the set of self-avoiding curves in $\mathscr{D}$. The collection of families (finite or countable) of curves in $\mathscr{D}(\mathscr{D}_s)$ will be denoted by $\mathscr{D}^*(\mathscr{D}_s^*)$.

Clearly, any $\gamma \in \mathscr{D}_s$ splits the square Q into two disjoint connected subsets denoted by A_γ, B_γ; we set $V(\gamma) = \min\{|A_\gamma|, |B_\gamma|\}$. We also define, for any curve $\gamma \in \mathscr{D}$, the Wulff functional $W(\gamma)$ as

$$W(\gamma) = \int_\gamma \tau(\vec{n}_s)\, ds$$

where s is the length parameter of the curve γ and $\tau(\vec{n}_s)$ is the surface tension at inverse temperature β in the direction of the normal $\vec{n}_s$ to the curve γ at the point s. Finally, we define the function $\bar{\varphi}(v)$, $0 < v \leqslant \infty$, as

$$\bar{\varphi}(v) \equiv \begin{cases} \inf_{\gamma \in \mathscr{D}_s:\, V(\gamma) = v} W(\gamma) & \text{if } 0 < v \leqslant \frac{1}{2} \\ \bar{\varphi}(1/2) & \text{if } \frac{1}{2} < v \end{cases} \tag{5.1}$$

$\bar{\varphi}(v)$ can be computed exactly [see ref. 17, where, however, there is a mistake in the expression for $\bar{\varphi}(v)$ due to a misprint]

$$\bar{\varphi}(v) = \begin{cases} \frac{1}{2}w\sqrt{v} & \text{if } \ 0 < v \leqslant v_0 \\ \tau(0) & \text{if } \ v_0 < v \end{cases} \tag{5.2}$$

where the constant w is the value of the Wulff functional $W(\gamma)$ on the Wulff curve $\gamma_F \in \mathbb{R}^2$ enclosing a unit area (see, e.g., ref. 17 or ref. 2) and the singularity point v_0 satisfies the equation

$$\tfrac{1}{2}w\sqrt{v_0} = \tau(0)$$

Notice that if for any $m \in [0, m^*]$ we set $V(m) = (m^* - m)/(2m^*)$, then $\bar{\varphi}(V(m)) = \varphi(m)$, where $\varphi(m)$ has been defined in Proposition 2.3.

For future purposes it is convenient generalize a little bit the definition of $\bar{\varphi}(v)$ by showing that the infimum in (5.1) can be taken not only over a single curve, but over a family $\mathscr{G} \in \mathscr{D}^*$. With this goal in mind, given a family $\mathscr{G} \in \mathscr{D}^*$, we fix an arbitrary point $x_0 \in Q$ in such a way that x_0 does not lie on one of the curves of $\mathscr{G}$ and we define the set $A_{\mathscr{G}}$ as the union of those points $x \in Q$ such that any path connecting x with x_0 and intersecting in a finite number of points the curves in $\mathscr{G}$ intersects them an odd number of times counting multiplicity. We define the *phase volume* of the family $\mathscr{G}$, $V(\mathscr{G})$, as the minimum between $|A_{\mathscr{G}}|$ and $|Q \backslash A_{\mathscr{G}}|$. Using the above definitions, we have the following result.

Lemma 5.1. Let $W(\mathscr{G}) = \sum_i W(\gamma_i)$. Then, for any $0 < v \leqslant \frac{1}{2}$ the following holds:

$$\bar{\varphi}(v) = \inf_{\substack{\mathscr{G} \in \mathscr{D}^* \\ V(\mathscr{G}) \geqslant v}} W(\mathscr{G})$$

Proof. It is clearly sufficient to prove that for any $0 < v \leqslant \frac{1}{2}$

$$\bar{\varphi}(v) \leqslant \inf_{\substack{\mathscr{G} \in \mathscr{D}^* \\ V(\mathscr{G}) \geqslant v}} W(\mathscr{G})$$

Given a family $\mathscr{G}$, we assume, without loss of generality, that its phase volume coincides with $|A_{\mathscr{G}}|$, so that $|A_{\mathscr{G}}| \leqslant \frac{1}{2}$. Let now $\{A^\alpha\}$ be the decomposition of the set $A_{\mathscr{G}}$ into mutually disjoint connected components and let, for any given α, $\gamma_1^\alpha \cdots \gamma_{k_\alpha}^\alpha$, $k_\alpha \leqslant +\infty$, be a decomposition of the curve $\partial A^\alpha \cap Q$ into countably many self-avoiding curves in $\mathscr{D}$. Using the fact that $|A^\alpha| \leqslant |A_{\mathscr{G}}| \leqslant \frac{1}{2}$, it is not difficult to check that

$$\sum_{i=1}^{k_\alpha} V(\gamma_i^\alpha) \geqslant |A^\alpha|$$

so that

$$\sum_{\alpha} \sum_{i=1}^{k_\alpha} V(\gamma_i^\alpha) \geqslant |A_{\mathscr{G}}| = V(\mathscr{G}) \geqslant v$$

Moreover, it is obvious that

$$W(\mathscr{G}) = \sum_{\alpha} \sum_{i=1}^{k_\alpha} W(\gamma_i^\alpha)$$

Thus we immediately conclude that

$$\inf_{\substack{\mathscr{G} \in \mathscr{D}^* \\ V(\mathscr{G}) \geqslant v}} W(\mathscr{G}) \geqslant \inf_{\substack{\mathscr{G} \in \mathscr{D}_s^* \\ \sum_i V(\gamma_i) \geqslant v}} W(\mathscr{G}) \tag{5.3}$$

We observe at this point that, because of (5.1), (5.2), one has

$$W(\gamma) \geqslant \bar{\varphi}(V(\gamma)) \qquad \forall \gamma \in \mathscr{D}_s$$

and

$$\sum_{i=1}^{\infty} \varphi(v_i) \geqslant \varphi\left(\sum_{i=1}^{\infty} v_i\right) \qquad \forall \{v_i\}_{i=1}^{\infty}$$

Thus the RHS of (5.3) can be bounded from below by

$$\inf_{\substack{\gamma_1 \cdots \gamma_n \cdots ;\, \gamma_i \in \mathscr{D}_s \\ \sum_i V(\gamma_i) \geqslant v}} \sum_i \bar{\varphi}(V(\gamma_i)) \geqslant \bar{\varphi}\left(\sum_i V(\gamma_i)\right) \geqslant \bar{\varphi}(v)$$

and the lemma follows. ∎

5.2. Proof of Proposition 2.3

The proof of this result follows closely the proof given by Ioffe[8] for the case of + boundary conditions. Before starting, we need to introduce some more notation.

In the following Λ will always denote the square Q_L. and for simplicity we let $N = 2L + 1$ be the length of its sides.

We then choose two real numbers $0 < \nu < b < \frac{1}{4}$ and we say that a contour Γ is b-large either if it is closed surrounding an area larger than L^{2b}, or if it is open and it splits the set Q_L into two parts each of which has an area larger than L^{2b}. Here we adopt the convention that the area of a finite set $A \subset \mathbb{Z}^2$ is simply the number of sites of A. The subscript b in objects like $C_b(\cdot)$, $C_b^*(\cdot)$, $\mathscr{G}_{A,b}^{\varnothing}(\sigma)$ means that we consider only b-large contours.

Given a sequence $(u_1,..., u_n)$ of dual sites in Λ^*, we denote by $P = [u_1,..., u_n]$ the polygon

$$[u_1,..., u_n] = \bigcup_{i=2}^{n} [u_{i-1}, u_i]$$

If $u_1 = u_n$, P is said to be closed ($\delta P = \varnothing$), otherwise it is open and its boundary is $\delta P = \{u_1, u_n\}$. We call P a *ν-skeleton* if

$$\frac{L^\nu}{12} \leqslant |x_{i+1} - x_i|_2 \leqslant L^\nu \qquad \forall i = 1,..., n-1$$

Given now a contour γ, we say that γ is *consistent* with a ν-skeleton $P = [u_1,..., u_n]$ and we write $\gamma \sim P$ if:

(i) $\delta\gamma = \delta P$.

(ii) The vertices of P lie on γ in the order corresponding to the natural order of the vertices of γ.

(iii) The Hausdorff distance between any edge $l_i \equiv [x_{i-1}, x_i]$ of P and the part of γ connecting x_{i-1} with x_i is not greater than L^ν.

Using the construction of Lemma 5.11 in ref. 2, is easy to check that, for any L large enough and any contour γ with $\text{diam}(\gamma) \geqslant L^\nu$, there always exists a ν-skeleton S consistent with γ. In particular, for any L large enough, it will always be possible to associate a particular ν-skeleton S to any b-large contour γ. We assume that a definite choice has been made once and for all, and $\hat{S}(\gamma)$ denotes *the* skeletons of γ. Finally, given a set $\mathscr{S} = \{S_1,..., S_r\}$ of skeletons, we denote by $E(\mathscr{S})$ the set of all configurations $\sigma \in \Omega_\Lambda$ such that the family of the ν-skeletons of their b-large contours coincides with $\mathscr{S}$. We also set

$$W(\mathscr{S}) = \sum_{i=1}^{r} W(S_i)$$

where, for any ν-skeleton $S = [x_1,..., x_k]$,

$$W(S) \equiv \sum_{i=2}^{k} \tau(x_i - x_{i-1})$$

Notice that, for a given ν-skeleton S, the number k of vertices of S and the value of the Wulff functional $W(S)$ are related by the following simple inequality, which follows from the definition of $W(S)$ and from (1.12):

$$W(S) \geqslant \frac{\tau(0)\, L^{\nu}}{20}\, k \tag{5.4}$$

Given $0 < d < \varphi(m)$, we define the event

$$K \equiv \{\sigma \colon W(\hat{S}(\sigma)) \geqslant (\varphi(m) - d)\, L\}$$

where $\hat{S}(\sigma)$ denotes the collection of the ν-skeletons associated to the b-large contours of σ. Clearly, $\hat{S}(\sigma)$ can also be the empty set. Then we write

$$\mu_{\Lambda}^{\beta, \varnothing}\{m_{\Lambda}(\sigma) = m\} \leqslant \mu_{\Lambda}^{\beta, \varnothing}\{m_{\Lambda}(\sigma) = m \mid K^{c}\} + \mu_{\Lambda}^{\beta, \varnothing}(K) \tag{5.5}$$

The second term of (5.5) can bounded from above by adapting to our case the technique of Section 10 of ref. 16. More precisely, we have the following result.

Lemma 5.2. For any $\beta > \beta_c$

$$\liminf_{L \to \infty} -\frac{1}{\beta N} \log \mu_{\Lambda}^{\beta, \varnothing}(K) \geqslant \varphi(m) - d$$

Proof. We write

$$\mu_{\Lambda}^{\beta, \varnothing}(K) \leqslant \sum_{\substack{\mathscr{S} = \{S_1, \dots, S_n\} \\ W(\mathscr{S}) \geqslant (\varphi(m) - d)\, L}} \mu_{\Lambda}^{\beta, \varnothing}(E(\mathscr{S})) \tag{5.6}$$

In order to estimate the generic term $\mu_{\Lambda}^{\beta, \varnothing}(E(\mathscr{S}))$, we will use the following lemma (compare with Lemma 10.1 in ref. 16, where + boundary conditions are considered):

Lemma 5.3. Let $\mathscr{S} = \{S_1, \dots, S_k\}$ be a collection of ν-skeletons. For any $\beta > \beta_c$

$$\mu_{\Lambda}^{\varnothing}(E(\mathscr{S})) \leqslant e^{-\beta W(\mathscr{S})}$$

Before proving Lemma 5.3, we finish the proof of Lemma 5.2. For each collection of skeletons $\mathscr{S}$, we let $N(\mathscr{S})$ be the sum of the number of vertices of all skeletons in $\mathscr{S}$, and A_k is the number of $\mathscr{S}$ with $N(\mathscr{S}) = k$.

Only terms with $N(\mathscr{S}) \leqslant 2\,|\Lambda^*|$ contribute to the RHS of (5.6), so using (5.4) and Lemma 5.3, we have, for all $\varepsilon \in (0, 1)$,

$$\text{RHS of } (5.6) \leqslant \exp[-(1-\varepsilon)\,\beta(\varphi(m)-d)\,L] \sum_{k=1}^{9L^2} A_k \exp\left(-\varepsilon \frac{\beta\tau_\beta L^v k}{20}\right)$$

A very rough estimate of A_k is given by

$$A_k \leqslant (2L+2)^{2k}\, k^k k!$$

where the first term takes into account the choices of the vertices in Λ^*, the second term is a bound on the number of possible partitions, and the last term accounts for the different arrangements. Since $k \leqslant 9L^2$, we have

$$A_k \leqslant (9L^2)^{3k} \leqslant e^{10k \log L}$$

which implies

$$\text{RHS of } (5.6) \leqslant \exp[-(1-\varepsilon)\,\beta(\varphi(m)-d)\,L] \sum_{k=1}^{9L^2} \exp\left[-\left(\frac{\varepsilon\beta\tau_\beta L^v}{20} - 10\log L\right)k\right]$$

Since ε can be arbitrarily small, we have

$$\liminf_{L\to\infty} -\frac{1}{\beta N} \log \mu^{\beta,\varnothing}_{Q_L}(K) \geqslant \varphi(m) - d$$

We are then left with the following.

Proof of Lemma 5.3. To simplify the notations, we consider the case of a single v-skeleton $S = [x_1, ..., x_n]$. The proof relies on the following two inequalities:

$$\mu^{\beta,\varnothing}_\Lambda(E(S)) \leqslant \sum_{\substack{\gamma \in C(\mathscr{E}^*_\Lambda, \delta S) \\ \gamma \sim S}} \frac{\tilde{Z}(\mathscr{E}^*_\Lambda \backslash \Delta(\gamma))}{\tilde{Z}(\mathscr{E}^*_\Lambda)} \tag{5.7}$$

and, letting $S' = [x_1, ..., x_{n-1}]$,

$$\sum_{\substack{\gamma \in C(\mathscr{E}^*_\Lambda, \delta S) \\ \gamma \sim S}} \tilde{Z}(\mathscr{E}^*_\Lambda \backslash \Delta(\gamma)) \leqslant \mathbf{E}^{\beta,\varnothing}_{\Lambda^*}(\sigma(x_{n-1})\,\sigma(x_n)) \sum_{\substack{\gamma \in C(\mathscr{E}^*_\Lambda, \delta S') \\ \gamma \sim S'}} \tilde{Z}(\mathscr{E}^*_\Lambda \backslash \Delta(\gamma)) \tag{5.8}$$

The proof of (5.8) is almost identical to the proof of Lemma 3.2, so we omit it. Iterating this inequality and using (1.10), one gets Lemma 5.3 if (5.7) holds. To prove (5.7), we let $\hat{\mathscr{E}}_\Lambda = \mathscr{E}^*_\Lambda \backslash \delta\Lambda$ and we have

$$\begin{aligned} \mu_\Lambda^{\beta,\varnothing}(E(S)) &\leqslant \mu_\Lambda^{\beta,\varnothing}\{\sigma \in \Omega_\Lambda : \exists \gamma \in \mathscr{G}^{\varnothing}_{\Lambda,b}, \hat{S}(\gamma) = S\} \\ &\leqslant \sum_{\substack{\gamma \in C(\hat{\mathscr{E}}_\Lambda, \delta S) \\ \gamma \sim S}} \mu_\Lambda^{\beta,\varnothing}\{\gamma \in \mathscr{G}^{\varnothing}_\Lambda(\sigma)\} \end{aligned} \tag{5.9}$$

Then we set $J = J_0^{\square}(\Lambda)$ and, using Proposition 1.3, we write

$$\begin{aligned} \mu_\Lambda^{\beta,\varnothing}\{\gamma \in \mathscr{G}^{\varnothing}_\Lambda(\sigma)\} &\leqslant \sum_{\lambda' \in C(\mathscr{E}^*_\Lambda \backslash \Delta''(\gamma), \delta\gamma)} \mu_\Lambda^{\beta,\varnothing}\{\gamma \cup \lambda' \in \mathscr{G}^+_\Lambda(\sigma)\} \\ &= \sum_{\lambda' \in C(\mathscr{E}^*_\Lambda \backslash \Delta''(\gamma), \delta\gamma)} \mu_\Lambda^{\beta,J,+}\{\gamma \cup \lambda' \in \mathscr{G}^+_\Lambda(\sigma)\} \end{aligned} \tag{5.10}$$

Moreover, if we let $\lambda = \gamma \cup \lambda'$, we have

$$\mu_\Lambda^{\beta,J,+}\{\lambda \in \mathscr{G}^+_\Lambda(\sigma)\} = w(\gamma)\, w_J(\lambda')\, \frac{\tilde{Z}^J(\mathscr{E}^*_\Lambda \backslash \Delta(\lambda))}{\tilde{Z}^J(\mathscr{E}^*_\Lambda \backslash \Delta''(\gamma))}\, \frac{\tilde{Z}^J(\mathscr{E}^*_\Lambda \backslash \Delta''(\gamma))}{\tilde{Z}^J(\mathscr{E}^*_\Lambda)}$$

(the weight of γ has no J superscript, since γ has no edges on $\delta\Lambda$). Even though in general $\Delta(\lambda) \supset \Delta''(\gamma) \cup \Delta(\lambda')$, it is easy see that

$$\tilde{Z}^J(\mathscr{E}^*_\Lambda \backslash \Delta(\lambda)) = \tilde{Z}^J(\mathscr{E}^*_\Lambda \backslash (\Delta''(\gamma) \cup \Delta(\lambda')))$$

Using the identity (remember that $\delta\gamma = \{x_1, x_n\}$)

$$\begin{aligned} &\sum_{\lambda' \in C(\mathscr{E}^*_\Lambda \backslash \Delta''(\gamma), \delta\gamma)} w_J(\lambda')\, \frac{\tilde{Z}^J(\mathscr{E}^*_\Lambda \backslash (\Delta''(\gamma) \cup \Delta(\lambda')))}{\tilde{Z}^J(\mathscr{E}^*_\Lambda \backslash \Delta''(\lambda))} \\ &\quad = \mathbf{E}^{\beta,J,+}_{\Lambda, \mathscr{E}^*_\Lambda \backslash \Delta''(\gamma)}(\sigma(x_1)\sigma(x_n)) \end{aligned} \tag{5.11}$$

and the fact that the above quantity is less than or equal to 1, we arrive at

$$\mu_\Lambda^{\beta,\varnothing}(E(S)) \leqslant \sum_{\substack{\gamma \in C(\hat{\mathscr{E}}_\Lambda, \{x_1, x_n\}) \\ \gamma \sim S}} w(\gamma)\, \frac{\tilde{Z}^J(\mathscr{E}^*_\Lambda \backslash \Delta''(\gamma))}{\tilde{Z}^J(\mathscr{E}^*_\Lambda)} \tag{5.12}$$

Thanks to Proposition 1.2, we know that we can replace J with $J = 1$ in the RHS of (5.12), obtaining an upper bound. In order to complete the proof of (5.7), one has to check that

$$\sum_{\substack{\gamma \in C(\hat{\mathscr{E}}_\Lambda, \{x_1, x_n\}) \\ \gamma \sim S}} w(\gamma)\, \tilde{Z}(\mathscr{E}^*_\Lambda \backslash \Delta''(\gamma)) = \sum_{\substack{\gamma \in C(\mathscr{E}^*_\Lambda, \{x_1, x_n\}) \\ \gamma \sim S}} w(\gamma)\, \tilde{Z}(\mathscr{E}^*_\Lambda \backslash \Delta(\gamma)) \tag{5.13}$$

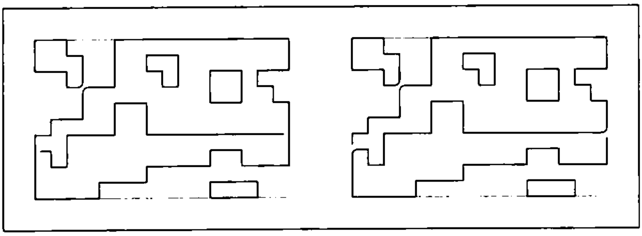

Fig. 4. Changing the splitting rules.

This equality follows from the existence of a bijective weight-preserving mapping from the set

$$\{(\gamma, \underline{\lambda}): \gamma \in C(\hat{\mathscr{E}}_\Lambda, \{x_1, x_n\}), \underline{\lambda} \in C^*(\mathscr{E}^*_\Lambda \backslash \Delta''(\gamma))\}$$

onto the set

$$\{(\gamma', \underline{\lambda}'): \gamma' \in C(\mathscr{E}^*_\Lambda, \{x_1, x_n\}), \underline{\lambda}' \in C^*(\mathscr{E}^*_\Lambda \backslash \Delta(\gamma'))\}$$

Such a mapping is obtained as follows (see also Fig. 4).

Let X be the set of all edges contained either in γ or in one of the contours $\lambda \in \underline{\lambda}$. Then X is a set of edges with $\delta X = \{x_1, x_n\}$. Split now X according to the splitting rules. The outcome is a collection of contours one of which is open with boundary $\{x_1, x_n\}$, while the rest are closed. We let γ' be the unique open contour and $\underline{\lambda}'$ the family of closed contours obtained. One can check that this is actually a bijective mapping between the two sets given above.

This completes the proof of Lemma 5.3. ∎

In order to finish the proof of Proposition 2.3, it is now sufficient to show that

$$\liminf_{N \to \infty} -\frac{1}{\beta N} \log \mu_\Lambda^{\beta, \varnothing}\{m_\Lambda(\sigma) = m \mid K^c\} = +\infty \tag{5.14}$$

We then denote by R the set of all $\underline{\gamma} \in C^*(\hat{\mathscr{E}}_\Lambda, \subset \mathscr{V}^c(\delta\Lambda))$ such that $W(\hat{S}(\underline{\gamma})) < (\varphi(m) - d)\, L$ and we write

$$\mu_\Lambda^{\beta, \varnothing}\{m_\Lambda(\sigma) = m \mid K^c\} \leqslant \sup_{\underline{\gamma} \in R} \mu_\Lambda^{\beta, \varnothing}\{m_\Lambda(\sigma) = m \mid \mathscr{G}^{\varnothing}_{\Lambda, b}(\sigma) = \underline{\gamma}\} \tag{5.15}$$

Bounds on (5.15) when $\underline{\gamma} \neq \varnothing$. We postpone to the end the analysis of the case $\underline{\gamma} = \varnothing$. If $\underline{\gamma} \neq \varnothing$, we can split Λ as a union $\Lambda = \bar{A}(\underline{\gamma}) \cup \bar{B}(\underline{\gamma})$ as follows: let $\underline{\lambda}$ and $\underline{\vartheta}$ be the two elements in $C^*(\mathscr{E}^*_\Lambda)$ which are mapped into

$\underline{\gamma}$ by (1.22). Order now the elements of $C^*(\mathscr{E}_\Lambda^*)$ is some arbitrary way and, assuming that $\underline{\lambda}$ precedes $\underline{\vartheta}$ in this order, define [see (1.16)]

$$\bar{A}(\underline{\gamma}) = \text{int}\, \underline{\lambda}, \qquad \bar{B}(\underline{\gamma}) = \Lambda \backslash \bar{A}(\underline{\gamma}) = \text{int}\, \underline{\vartheta}$$

Let also [see (1.15)]

$$\Delta_A^s(\underline{\gamma}) = \Delta^s(\underline{\gamma}) \cap \bar{A}(\underline{\gamma}), \qquad \Delta_B^s(\underline{\gamma}) = \Delta^s(\underline{\gamma}) \cap \bar{B}(\underline{\gamma})$$

and

$$A(\underline{\gamma}) = \bar{A}(\underline{\gamma}) \backslash \Delta_A^s(\underline{\gamma}), \qquad B(\underline{\gamma}) = \bar{A}(\underline{\gamma}) \backslash \Delta_B^s(\underline{\gamma})$$

Then we have the following result.

Lemma 5.4. There exist $d' = d'(\beta, d) > 0$ and $L_0(\beta, d)$ such that for any $L > L_0$ and for any $\underline{\gamma} \in R$ with $\underline{\gamma} \neq \varnothing$, if we let $J = J_0^\square(\Lambda)$, we have

$$\begin{aligned} &\mu_\Lambda^{\beta, \varnothing}\{m_\Lambda(\sigma) = m \mid \mathscr{G}_{\Lambda, b}^{\varnothing}(\sigma) = \underline{\gamma}\} \\ &\quad \leqslant \chi\{|A| \geqslant d'N^2\}\, \mu_{A, b}^{\beta, J, +}\{|m_A(\sigma) - m^*| \geqslant d'\} \\ &\quad\quad + \chi\{|B| \geqslant d'N^2\}\, \mu_{B, b}^{\beta, J, +}\{|m_B(\sigma) - m^*| \geqslant d'\} \end{aligned} \tag{5.16}$$

where the subscript b means that the measure is conditioned not to have any large contour, and we have set $A = A(\underline{\gamma})$, $B = B(\underline{\gamma})$.

Remark. The couplings J have the effect that the $+$ boundary conditions act only on $\delta A \backslash \delta \Lambda$, while we have free boundary conditions on $\delta A \cap \delta \Lambda$ (similarly for B).

Proof. We assume that L and $\underline{\gamma}$ have been chosen satisfying the hypothesis of the lemma, and we let $\Delta^s = \Delta^s(\underline{\gamma})$ and so on. Let $\mathscr{S} = S(\underline{\gamma})$ be the set of the skeletons of the large contours. Since $W(\mathscr{S}) \leqslant (\varphi(m) - d)\, L$, the volume of Δ^s satisfies

$$|\Delta^s| \leqslant CN^{1+2\nu} \tag{5.17}$$

for a suitable constant C independent of L.[19] Moreover, using Lemma 5.1, (5.3), and the fact that $\varphi(m) = \bar{\varphi}((m^* - m)/(2m^*))$, we get that there exists $\varepsilon(d) > 0$ such that the phase volume $V(\mathscr{S})$ is bounded by

$$V(\mathscr{S}) \leqslant (1 - \varepsilon(d)) \left(\frac{m^* - m}{2m^*} \right) N^2 \tag{5.18}$$

for any L large enough. Finally, we assume, without loss of generality, that

$$|\bar{A}| \leqslant \tfrac{1}{2} N^2$$

Then, using Lemma 5.13 of ref. 2 and (5.17), we get that

$$||A| - V(\mathscr{S})| \leqslant C_1 N^{1+2\nu}$$

for a suitable constant C_1, so that, using (5.18),

$$|A| \leqslant \left(1 - \frac{\varepsilon(d)}{2}\right)\left(\frac{m^* - m}{2m^*}\right) N^2 \qquad \text{and} \qquad |B| > \frac{1}{2} N^2 \tag{5.19}$$

for any N large enough.

Fix $0 < d' \ll 1$ and let us distinguish between the two cases (i) $|A| \leqslant d'N^2$ and (ii) $|A| \geqslant d'N^2$.

In the first case, if we denote by $m_B(\sigma)$ the (normalized) magnetization inside B, we get

$$|m_B(\sigma) - m| \leqslant 2CN^{2\nu-1} + 2d'$$

and the lemma follows for d'small enough, since $2\nu < \frac{1}{2}$.

In the second case, let us suppose that

$$||m_B(\sigma)| - m^*| \leqslant d' \qquad \text{and} \qquad ||m_A(\sigma)| - m^*| \leqslant d'$$

Then, using (5.19) and the fact that the total magnetization is equal to $m \geqslant 0$, it is clear that $m_B \approx m^*$ and $m_A \approx -m^*$ for d' small enough, so that

$$\left| |A| - \frac{(m^* - m)}{2m^*} N^2 \right| \leqslant C_2 d' + 2CN^{2\nu-1}$$

for a suitable constant C_2. Thus, by taking d' small enough independently of L, we get a contradiction with (5.19). The lemma follows from the trivial estimate

$$\mu_A^{\pm, b}\{||m_A| - m^*| \geqslant d'\} \leqslant \mu_A^{+, b}\{|m_A - m^*| \geqslant d'\} \quad \blacksquare$$

We are left with the estimate of the two terms in the RHS of (5.16), that is, thanks to (5.17), with the estimate of

$$\sup_{\substack{A \subset Q_L \\ dN^2 \leqslant |A| \leqslant N^2 - N^{2b},\ |\partial^+ A| \leqslant CN^{1+2\nu}}} \mu_{A, b}^{\beta, J, +}\{|m_A(\sigma) - m^*| \geqslant d\} \tag{5.20}$$

where $J = J_0^{\square}(A)$, so that the + boundary conditions act only on $\delta A \backslash \delta A$. Let, for any $\sigma \in \Omega_A$,

$$\mathscr{C}(\sigma) = \{x \in A : \exists \text{ a path from } x \text{ to } \partial A \text{ inside } A$$
$$\text{such that } \sigma(y) = -1 \text{ along the path}\}$$

Notice that if $u \in A$ is such that $d(u, A^c) \geqslant N^{2b}$, then $\mu^{\beta,J,+}_{A,b}$-almost surely, $u \notin \mathscr{C}(\sigma)$, since otherwise there would be a cluster of minus spins inside A with area larger than N^{2b} and, by consequence, there would be a large contour inside A. Hence

$$|\mathscr{C}(\sigma)| \leqslant 4N^{1+2b} \qquad \mu^{\beta,J,+}_{A,b}\text{-almost surely} \tag{5.21}$$

We then denote by $\mathscr{C}^*$ the collection of all $\mathscr{C} \subset A$ such that $\mu^{\beta,J,+}_{A,b}\{\mathscr{C}(\sigma) = \mathscr{C}\} > 0$ and we write

$$\begin{aligned} &\mu^{\beta,J,+}_{A,b}\{|m_A(\sigma) - m^*| \geqslant d\} \\ &\quad \leqslant \sup_{\mathscr{C} \in \mathscr{C}^*} \mu^{\beta,J,+}_{A,b}\{|m_A(\sigma) - m^*| \geqslant d \,|\, \mathscr{C}(\sigma) = \mathscr{C}\} \end{aligned} \tag{5.22}$$

Given now $\mathscr{C} \in \mathscr{C}^*$, let $\bar{\mathscr{C}} = \mathscr{C} \cup \partial^+\mathscr{C} \cup \partial Q_L$ and let $A' = A \setminus \bar{\mathscr{C}}$. By construction we have that any $\sigma \in \Omega_{A'}$ such that $\mathscr{C}(\sigma) = \mathscr{C}$ satisfies $\sigma(x) = +1$ $\forall x \in \partial^+ A'$. Moreover, thanks to (5.21), we have that

$$|A'| \geqslant \frac{3d}{4} N^2$$

for any N large enough, so that

$$|m_{A'}(\sigma) - m_A(\sigma)| \geqslant d/2$$

for any N large enough. Thus we can bound from above a generic term in the RHS of (5.22) by

$$\begin{aligned} &\mu^{\beta,J,+}_{A,b}\{|m_A(\sigma) - m^*| \geqslant d \,|\, \mathscr{C}(\sigma) = \mathscr{C}\} \\ &\quad \leqslant \mu^{\beta,+}_{A',b}\left\{|m_{A'}(\sigma) - m^*| \geqslant \frac{d}{2}\right\} \\ &\quad \leqslant \mu^{\beta,+}_{A',b}\left\{m_{A'}(\sigma) \leqslant m^* - \frac{d}{2}\right\} + \frac{\mu^{\beta,+}_{A'}\{m_{A'}(\sigma) \geqslant m^* + d/2\}}{\mu^{\beta,+}_{A'}\{\text{all contours are small}\}} \end{aligned} \tag{5.23}$$

Using Lemma 3.1 of ref. 8 (see the appendix of ref. 18 for a simplified proof), we have

$$\mu^{\beta,+}_{A',b}\left\{m_{A'}(\sigma) \leqslant m^* - \frac{d}{2}\right\} \leqslant C\exp(-CN^{2-4b}) \tag{5.24}$$

for a suitable positive contant C. To bound the other term, since

$$|\partial A'| \leqslant |\partial A| + |\bar{\mathscr{C}}| \leqslant CN^{1+2v} + 20N^{1+4b} + 4N \leqslant N^{\gamma}, \qquad \gamma < 2$$

for any N large enough, we can use inequality (4.8) in ref. 8 and we get

$$\mu_{A'}^{\beta,+}\left\{m_{A'}(\sigma)\geqslant m^*+\frac{d}{2}\right\}\leqslant C_1\exp(-C_1N^2) \tag{5.25}$$

for any N large enough for a suitable constant C_1. We finally estimate the denominator in (5.23). We have

$$\begin{aligned}\mu_{A'}^{\beta,+}&\{\text{all contours are small}\}\\ &\geqslant 1-\mu_{A'}^{\beta,+}\{\text{there exists a } *\text{-path of } - \text{ spins with diameter}\geqslant L^b\} \quad (5.26)\\ &\geqslant 1-\mu^{\beta,+}\{\text{there exists a } *\text{-path of } - \text{ spins with diameter}\geqslant L^b\} \quad (5.27)\end{aligned}$$

where we have used the FKG property in the second inequality and $\mu^{\beta,+}$ denotes the infinite-volume plus phase. Finally, using the results of ref. 1, we have that the RHS of (5.27) tends to zero as $N\to\infty$. If we now combine (5.24)–(5.27), we get that

$$\liminf_{N\to\infty}-\frac{1}{\beta N}\log\mu_{\Lambda}^{\beta,\varnothing}\{m_\Lambda(\sigma)=m\mid K^c\} = +\infty$$

and Proposition 2.3 follows. ∎

Bounds on (5.15) when $\underline{\gamma}=\varnothing$. We finally estimate

$$\mu_{\Lambda,b}^{\beta,\varnothing}\{m_\Lambda(\sigma)=m\mid\mathscr{G}_b=\varnothing\} \tag{5.28}$$

For this purpose let $\bar{\Lambda}=\{x\in\Lambda: d(x,\partial\Lambda)\geqslant N^{2b}\}$. Then we have that $\mu_{\Lambda}^{\beta,\varnothing}$ almost surely there exists a closed $*$-path $\mathscr{C}\subset\Lambda\backslash\bar{\Lambda}$ encircling $\bar{\Lambda}$ such that either $\sigma=+1$ on $\mathscr{C}$ or $\sigma=-1$ on $\mathscr{C}$. This is so because on the contrary there would be two paths of spins of opposite sign connecting $\partial\Lambda$ with $\partial\bar{\Lambda}$. But this implies the existence of a b-large contour between them. Given now a closed $*$-path $\mathscr{C}\subset\Lambda\backslash\bar{\Lambda}$ encircling $\bar{\Lambda}$, let A be the region enclosed by $\mathscr{C}$. Since by construction $|m_\Lambda(\sigma)-m_A(\sigma)|\leqslant 4N^{2b-1}$, we get

$$(5.28)\leqslant\sup_{\tau=\pm}\ \sup_{\substack{A\subset\Lambda\text{ connected}\\ A\supset\bar{\Lambda}}}\mu_{A,b}^{\beta,\tau}\{|m_A(\sigma)-m|\leqslant 4N^{2b-1}\} \tag{5.29}$$

We can now proceed as in the previous case and obtain

$$\liminf_{N\to\infty}-\frac{1}{\beta N}\log\left[\sup_{\tau=\pm}\ \sup_{\substack{A\subset\Lambda\text{ connected}\\ A\supset\bar{\Lambda}}}\mu_{A,b}^{\beta,\tau}\{|m_A(\sigma)-m|\leqslant 4L^{2b-1}\}\right]=+\infty \quad ∎$$

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REFERENCES

1. J. Chayes, L. Chayes, and R. H. Schonmann, Exponential decay of connectivity in the two-dimensional Ising model, *J. Stat. Phys.* **49**:433 (1987).
2. R. Dobrushin, R. Kotecký, and S. Shlosman, *Wulff Construction. A Global Shape from Local Interaction* (American Mathematical Society, Providence, Rhode Island, 1992).
3. R. L. Dobrushin and S. Shlosman, Constructive criterion for the uniqueness of Gibbs fields, in *Statistical Physics and Dynamical Systems*, Fritz, Jaffe, and Szász, eds. (Birkhauser, Boston, 1985), p. 347.
4. C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre, Correlation inequalities on some partially ordered sets, *Commun. Math. Phys.* **22**:89 (1971).
5. Y. Higuchi, Coexistence of infinite (*)-cluster II: Ising percolation in two dimension, *Prob. Theory Related Fields* **97**:1 (1993).
6. Y. Higuchi and N. Yoshida, Slow relaxation of 2D-stochastic Ising models at low temperature, preprint (1995).
7. D. Ioffe, Large deviation for the 2D Ising model: A lower bound without cluster expansion, *J. Stat. Phys.* **74**(1/2):411 (1994).
8. D. Ioffe, Exact large deviation bounds up to T_c for the Ising model in two dimensions, preprint (1994).
9. M. Jerrum and A. Sinclair, Polynomial-time approximation algorithms for the Ising model, *SIAM J. Comput.* **22**:1087 (1993).
10. T. M. Ligget, *Interacting Particles Systems* (Springer-Verlag, Berlin, 1985).
11. F. Martinelli, On the two dimensional dynamical Ising model in the phase coexistence region, *J. Stat. Phys.* **76**(5/6):1179 (1994).
12. E. Marcelli and F. Martinelli, Some new results on the 2d kinetic Ising model in the phase coexistence region, preprint (1995).
13. F. Martinelli and E. Olivieri, Approach to equilibrium of Glauber dynamics in the one phase region I: The attractive case, *Commun. Math. Phys.* **161**:447 (1994).
14. F. Martinelli, E. Olivieri, and R. H. Schonmann, For Gibbs state of 2D lattice spin systems weak mixing implies strong mixing, *Commun. Math. Phys.* **165**:33 (1994).
15. F. Martinelli, E. Olivieri, and E. Scoppola, Metastability and exponential approach to equilibrium for low temperature stochastic Ising model, *J. Stat. Phys.* **61**:1105 (1990).
16. C. E. Pfister, Large deviations and phase separation in the two-dimensional Ising model, *Helv. Phys. Acta* **64**:953 (1991).
17. S. B. Shlosman, The droplet in the tube: A case of phase transition in the canonical ensemble, *Commun. Math. Phys.* **125**:81 (1989).

18. B. Simon, Correlation inequalities and the decay of correlations in ferromagnets, *Commun. Math. Phys.* **77**:111 (1980).
19. R. H. Schonmann and S. B. Shlosman, Complete analiticity for 2D Ising model completed, *Commun. Math. Phys.* **170**:453 (1995).
20. R. H. Schonmann and S. B. Shlosman, Constrained variational problems with applications to the Ising model, preprint.
21. R. H. Schonmann and S. B. Shlosman, In preparation.